# SOME ASPECTS OF GRAVITATIONAL COLLAPSE IN MODIFIED GRAVITY

Thesis submitted for the degree of
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in Physics

Submitted by
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To

My late grandmother
SMT. KARUNA CHATTERJEE

and

My late uncle, friend, philosopher and guide
SRI ALOK BHATTACHARYYA.

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# Abstract

This thesis involves a study of different aspects of the phenomenon of gravitational collapse in modified theory of gravity. It is divided into three main parts. The first part of the thesis, which is the "**Introduction**", deals with a discussion of general relativity, modified gravity and gravitational collapse. The second part, which is titled "**Some Aspects of Gravitational Collapse in** $f(R,T)$ **Gravity**", deals with the crux of the research carried out in this thesis. The third and final part, the "**Summary and Outlook**" discusses the significance and limitations of these works, and possible future directions.

In the first part of the thesis, the "**Introduction**", we have begun with an overview of the phenomenon of gravitational collapse in the very first chapter. We have then proceeded to introduce the rudimentaries of General Relativity (GR) in the next chapter, and provided basic mathematical formulae which describe the spacetime curvature and its relation to the matter content present in that spacetime. The third chapter introduces the theories of modified gravity and the motivation behind these theories. The fourth chapter discusses the collapse formalism, and refers to the existing literature regarding collapse in GR, $f(R)$ and $f(R,T)$ gravity. This finally leads us to the final chapter of the "**Introduction**", which discusses briefly, the aspects of collapse investigated in this thesis, viz., dynamical stability, causal heat transport, formation of singularity and apparent horizon, and structure scalars.

The second part of the thesis, "**Some Aspects of Gravitational Collapse in** $f(R,T)$ **Gravity**", is divided into three chapters.

The first chapter of this part deals with the dynamical stability bounds for a most general spherically symmetric dissipative matter configuration in the framework of $f(R,T)$ gravity. For an $f(R,T)$ function which is quadratic in the Ricci scalar $R$, and linear in the Trace $T$, we have invoked the perturbation scheme and found the bounds on the adiabatic index for the Newtonian and the post-Newtonian regimes within which the collapsing system will retain its stability. We have utilised the theory of causal transport established by Muller, Israel and Stewart [126, 127, 128, 129] and established the validity of the weak equivalence principle and found the point at which there is a switch between expansion and collapse of the system.

The second chapter of this part deals with the collapse of an isotropic spherically symmetric fluid undergoing dissipation in the form of radial heat flow in $f(R,T)$ gravity described by an $f(R,T)$ function linear in both $R$ and $T$. Considering the exterior line-element to be that of the generalised Vaidya form [165], and the exterior matter to be a combination of Type-I and Type-II fluids, the junction conditions

of $f(R,T)$ gravity have been examined. The time of formation of singularity has been determined with the help of the condition of pressure isotropy. The time of apparent horizon formation has been obtained, and finally the difference between these two expressions for time has been examined to find the constraints that need to be valid for the formation of a black hole as the final end state of the collapse.

The third chapter of this part examines the role of structure scalars (constructed via an orthogonal splitting of the Riemann tensor) on the physical properties of the collapsing matter involving charge and undergoing dissipation, which is modelled by the most general relativistic fluid. These physical properties include the energy density inhomogeneity, the pressure anisotropy, heat dissipation, and the evolution of the expansion and shear of the collapsing matter. Since any physical property of a collapsing system, apart from isotropic pressure and homogeneous energy density, adds to the complexity of the system, a discussion on the complexity factor of the collapsing configuration have also been included. The geometry is once again, considered to be spherically symmetric. It is highlighted how the charge affects each of the structure scalars, and also the mass-energy content of the collapsing sphere. The entire treatment has been kept general, without assuming a form of the $f(R,T)$ function. Later, two special cases, one for the choice of an $f(R,T)$ function linear in both $R$ and $T$, and another case, for a relativistic dust ball have been considered. The $f(R,T)$ junction conditions have been presented, showing the matching conditions for the matter Lagrangian and their derivatives at the boundary. The energy conditions are also presented and the possibility of violation of the Strong Energy Condition has been discussed.

Finally, the third part of the thesis, the "**Summary and Outlook**", discusses the overall takeaways of the work presented in this thesis.

# List of Publications in this Thesis

This is a list of the journal versions and arXiv versions of the papers included in this thesis.

- **Title of the Paper :** "Dynamical Conditions and Causal Transport of Dissipative Spherical Collapse in $f(R,T)$ Gravity"

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# Declaration

As the author of this thesis, I confirm that no part of this thesis has been previously utilised as part of any other academic programme. The analyses in these works have been carried out solely by the author under the guidance of the supervisor. Rigorous checking of the manuscript has been done to eliminate errors as far as possible. Even then, some errors may still persist in the manuscript, for which the responsibility is borne solely by the author.

Uttaran Ghosh

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# Part I

# Introduction

# Chapter 1

# Overview and Objective of this Thesis

The revolutionary theory of General Relativity (GR), which was put forward by Albert Einstein in 1915 [1], and was a generalization of the Special Theory of Relativity formulated by him in 1905, tells us that the presence of matter and energy in a spacetime induces curvature in the geometry of the spacetime, which, in turn dictates the trajectory of matter particles and photons. It has helped in providing explanations to various astrophysical phenomena, such as the perihelion precession of Mercury and other planets, the bending of light path in the region surrounding a massive body, and the gravitational redshift.

The astrophysical phenomenon of gravitational collapse plays a very crucial role in the evolution of the universe. It is responsible for structure formation in the universe (like the formation of galaxies and clusters of galaxy), as well as the collapse of a massive stellar object under self-gravity, and the eventual point where this process reaches a finality at the end of the life cycle of the star. The inward gravitational attraction towards the centre of a stellar conglomeration of matter creates compression which raises the temperature inside the star to a high value sufficient to ignite nuclear fusion. The heat generated by nuclear fusion reaction gives rise to a thermal pressure which is directed outwards, and, as a result, counteracts the gravitational contraction of the matter. If the inward contraction and the outward thermal pressure exactly cancel out the effects of one another, the star remains in a state of stability. However, once all the fuel inside the star has undergone nuclear fusion, the outward thermal pressure ceases to exist any longer, and the gravitational contraction proceeds, until it becomes so much stronger that all the electrons inside the atoms of the star are brought closer together. This packing of electrons in a small volume generates what is known as the electron degeneracy pressure. For stars less than approximately 1.4 solar masses ($1.4M_{\odot}$), which is known as the Chandrasekhar Limit [60], this electron degeneracy pressure is sufficient to balance the gravitational contraction and the star reaches a stable state known as the white dwarf. For stars heavier than the Chandrasekhar Limit, the gravitational collapse cannot be counterbalanced by the electron degeneracy pressure and the collapse continues, with the result that neutrons are generated by the process of inverse beta decay. This results in a degenerate neutron gas which gives rise to the neutron degeneracy pressure. For stars less than a certain upper limit of mass, called the Tolman-Oppenheimer-Volkoff Limit, the neutron degeneracy pressure effectively opposes and balances the inward gravitational force, resulting in yet another stable configuration known as the neutron star. For still heavier stars, the neutron degeneracy pressure is not sufficient to resist the gravitational force, and a continued collapse occurs till the star reaches a final stage. The nature of the final state of collapse, where the physical properties of the star, like the density, and the curvature of the spacetime tend to diverge, and the laws of physics are no longer valid, is called a spacetime singularity. If the final singularity is visible to an external observer, it is known as a naked singularity, and if not, then it is a black hole. The nature of the final singularity, which is found to be dependent on the nature of the collapsing matter, is a primary focus of interest in the study of gravitational collapse.

From recent astronomical observations, based on observational data obtained from the Type-Ia supernova [3, 4], it has been concluded that the universe is undergoing a late-time accelerated expansion, which is dominated by a quantity called the dark energy [2], not much information about which is available till date. An assumption to explain this accelerating phase of evolution of the universe, is that General Relativity might not be a viable theory at extremely large scales. Also, the galactic rotation curves cannot be explained using GR. As a result, possibilities of other alternative gravity theories have come up over the years. One such family of theories, known as the $f(R)$ gravity, is obtained by replacing the Ricci scalar in the Einstein-Hilbert action of GR, by a function of the Ricci scalar, which gives rise to higher order terms of spacetime curvature. It has been seen that this particular class of theories can indeed explain the late-time accelerated expansion of the universe, with the higher order curvature terms acting as the dark energy source terms. In General Relativity, the Cosmological constant have to be inserted by hand to the gravitational field equations so as to explain the role of Dark Energy. But in modified gravity like the $f(R)$ theory, the Dark Energy terms arise automatically from the action. Further the rotation curves of galaxies can also be explained by this theory, and it has also been found to pass the solar system tests. A further generalisation of this theory is to involve the matter contribution in the action and replace the Ricci scalar by a function $f(R,T)$, which involves both the Ricci scalar $R$ and the trace $T$ of the energy-momentum tensor describing the matter content of the spacetime. The $f(R,T)$ theory of gravity has successfully explained the acceleration of the universe during early times (inflation of the universe), at late times (due to the presence of dark energy) as well as the rotation curves of the galaxies (the effect of dark matter).

It is meaningful to study gravitational collapse in the framework of $f(R,T)$ theory, since this theory is justified from the point of view of cosmology, as explained above. If we consider the large scale structure of the universe, the matter content is not distributed uniformly everywhere. Therefore, the effect of dark matter (DM) is different in different regions of the universe and this difference is reflected in the nature of gravitational collapse in a given region of the universe. DM is responsible for ushering in and increasing the rate of gravitational collapse, although DM by itself does not collapse under self-gravity as it does not interact with electromagnetic or other forces. In the case of dark energy (DE), the situation is different because DE affects all parts of the universe in the same way and is not dependent on the background.

This provided us with the motivation for investigating the various aspects of gravitational collapse, such as the dynamical conditions, causal heat transfer, the formation of singularity and apparent horizon, and the conditions of formation of black hole at the final state of collapse, in the context of $f(R,T)$ gravity, which have been analysed in detail in this thesis, thereby making further inroads in this field of research over similar studies carried out by earlier and other contemporary researchers.

# Chapter 2

# General Relativity

In 1915, Albert Einstein put forward his General Theory of Relativity in [1], which was a generalization of his Special Theory of Relativity. Special Relativity deals with the fact that all inertial frames of reference are equivalent, and the velocity of light, denoted by $c$, has the same value, $3\times10^8 m/s$ in all inertial frames. General Relativity (GR) tells us that contrary to the idea from the Newtonian theory of gravitation, gravity is not a force, but a natural outcome of the curvature of spacetime. The presence of matter and energy in the spacetime, causes the spacetime to warp, which, in turn, dictates the trajectories of the matter content. This idea extends to encompass all non-inertial frames of references as well.

## 2.1 The Principle of Covariance

An important underlying feature of GR, is that the principle of covariance be obeyed by all physical laws. In other words, the mathematical equations describing the laws of physics can be expressed in a form that is independent of the choice of the coordinate system.

## 2.2 The Equivalence Principle

The equivalence of all inertial reference frames with respect to any physical experiment confined to a single inertial frame of reference in Special Relativity, is generalized in General Relativity in the following manner : *the laws of physics are dictated by special relativity inside a freely falling non-rotating reference frame, locally.* This is known as the *principle of equivalence.* By "locally", we mean a small enough region of spacetime with minimal curvature.

What follows in the next few sections is a mathematical description of curvature, where we move away from the flat-space geometry of the Minkowski spacetime in Special Relativity, to a curved-space geometry in GR.

## 2.3 Mathematical Representation of the Essentials

We have carried out our research in the four-dimensional spacetime, and have followed the convention of using Greek indices to represent the coordinates of the four-dimensional spacetime, i.e., the coordinates are $\{x^\mu\}$, with $\mu = 0, 1, 2, 3$. Here, the timelike component is represented by 0. For the three-dimensional spatial coordinates, we have utilised Latin indices, i.e., the coordinates of the three-dimensional space are $\{x^i\}$, with $i = 1, 2, 3$. Combining these two conventions, we can represent any point in the four-dimensional spacetime as $x^\mu = (x^0, x^i)$. Any such point in the four-dimensional spacetime is known as an 'event' in special relativity.

We adopt the "Einstein summation convention" which says that any index which appears both as a contravariant and a covariant index, must be summed over. For example, the quantity $a_\alpha b^\alpha$ implies that there is a sum over the index $\alpha$. For this repetition of the same index as a subscript and a superscript, the summation sign can be dropped.

In special relativity, the flat Minkowski spacetime is denoted by the second-rank tensor $\eta_{\mu\nu}$. In GR, the curved spacetime is analogously expressed by the second-rank tensor $g_{\mu\nu}$, where the components of these tensor are functions of the spacetime coordinates $x_\mu$. The line element in a curved spacetime is described by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.1}$$

where it can be positive, negative or zero. In other words, the curved spacetime is part of a manifold which is pseudo-Riemannian in nature. The covariant derivative of a vector $v^\mu$ is given by the expression

$$\nabla_\alpha v^\mu = \partial_\alpha v^\mu + \Gamma^\mu_{\alpha\nu} v^\nu, \tag{2.2}$$

where $\Gamma^\mu_{\alpha\nu}$ are called the connection coefficients, or the Christoffel's symbol of the second kind. These are symmetric in the covariant indices for a torsionless manifold, and are defined by,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \tag{2.3}$$

It is a convention to express the covariant derivative with a semicolon, viz., $\nabla_\alpha v^\mu = v^\mu_{;\alpha}$, and the partial derivative with a comma, viz., $\partial_\alpha v^\mu = v^\mu_{,\alpha}$.

### 2.3.1 Riemann Curvature Tensor, Ricci Tensor and Ricci Scalar

The Riemann curvature tensor $R^\rho_{\mu\nu\alpha}$ is expressed as

$$R^\rho_{\mu\nu\alpha} = \partial_\nu \Gamma^\rho_{\mu\alpha} - \partial_\alpha \Gamma^\rho_{\mu\nu} + \Gamma^\sigma_{\mu\alpha}\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\alpha}. \tag{2.4}$$

A contraction of the Riemann tensor in the first and last indices yields the Ricci tensor $R_{\mu\nu}$. In other words,

$$R_{\mu\nu} = R^\rho_{\mu\nu\rho}, \tag{2.5}$$

which when further contracted, gives us the Ricci scalar $R$, as can be seen :

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.6}$$

### 2.3.2 The Einstein Tensor and the Field Equations of GR

The Einstein tensor is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.7}$$

Equating the Einstein tensor to the energy-momentum tensor $T_{\mu\nu}$, gives us the field equation of General Relativity, which relates spacetime curvature (denoted by the Einstein tensor) to matter (denoted by the energy-momentum tensor), and is expressed as

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.8}$$

where $\kappa = \frac{8\pi G}{c^4}$, with $G$ being the Newtonian Gravitational Constant, and $c$ being the velocity of light. In reduced units, we consider $8\pi G = c = 1$.

# Chapter 3

# Modified Theories of Gravity

General relativity is a highly successful theory on many fronts. Till date it has been able to provide successful explanations of phenomena such as the precession of perihelion of Mercury, gravitational redshift, and the bending of light during an eclipse.

In the framework of General Relativity, we can arrive at the Friedmann Equations from the Einstein Field Equations for a spacetime having homogeneity and isotropy, which can describe the evolution of our universe. However recent observations in astronomy has led us to conclude that at two points of time, the universe underwent an accelerated expansion. The first accelerating phase was prior to the radiation dominated era, and is known as inflation, while the current accelerating phase has begun after the matter dominated epoch. This ongoing accelerated expansion is caused by an unknown component called dark energy [2]. Observations of Type Ia Supernovae [3, 4] led to the consequence that the deceleration parameter was coming out to be negative, which signified an accelerated expansion of the universe.

In General Relativity, if no cosmological constant is taken into account, the Friedmann equation of acceleration [5] together with the Strong Energy Conditions give us the result that the acceleration is negative. In other words, there is a decelerated expansion. Hence, the introduction of the cosmological constant becomes essential if one has to explain the current accelerated expansion of the universe. However its value is extremely small to be of significant contribution to the vacuum energy. Also its energy density is of the same scale as that of the matter, only for a short period of time in the entire evolution of our Universe. So there is no reason as to why it should be so during the current epoch, as is the case. Hence it is because of these magnitude and coincidence problems that we intend to look for an alternative theory in which we can do away with the cosmological constant and still find a way to explain the accelerating expansion of the universe. Further, GR also fails to give an explanation of the galactic rotation curves, which are the variation of the orbital velocities of stars in a galaxy, with respect to the radial distances of the stars from the centre of the galaxy. All these prompted researchers to lookout for alternative theories of gravity. Of some of the possible alternative classes of theories of gravitation, the $f(R)$ and $f(R,T)$ theories of gravity are now discussed in the following sections.

## 3.1 $f(R)$ Theories

The field equations for General Relativity are obtained by extremising the Einstein-Hilbert action, whose Lagrangian involves the Ricci scalar $R$. This action, denoted by $S$, is expressed as

$$S = \int \left( \frac{1}{2\kappa} R + \mathcal{L}_m \right) \sqrt{-g} d^4x, \tag{3.1}$$

where, we have considered the $(-+++)$ convention to describe the metric signature. Here, $g$ is the determinant of the spacetime metric $g_{\mu\nu}$, and $\mathcal{L}_m$ is the matter Lagrangian due to any matter field that might be present.

The simplest modification of this action which leads us to the modified gravity theories, is to replace the Ricci scalar $R$ in the action by a function $f(R)$. This results in a modified Einstein-Hilbert action for a class of theories, known as the $f(R)$ theories of gravity. Possibly the earliest mention of such a modification was made by Buchdahl [6], where he used the notation $\phi(R)$ in stead of $f(R)$. Extensive discussions on $f(R)$ theories can be found in literature [2, 7, 8]. There are different formalisms of $f(R)$ gravity, for example, the metric formalism, which treats the action to be dependent only on the metric and not the connection coefficients, and the Palatini formalism [10], where the action is varied with respect to both the metric and the connection. Here the matter Lagrangian is assumed to be independent of the latter. There is also the metric-affine formalism where both the action and the matter Lagrangian are varied with respect to the metric and the connection. These different formalisms lead to different expressions for the field equations in $f(R)$ gravity, although all of them reduce to the Einstein field equations when $f(R) = R$, that is, in General Relativity. Solar system tests, such as the precession of perihelion of Mercury, the bending of light around massive objects, and the delay caused in the transmission of electromagnetic waves through a gravitational field, which is also known as the Shapiro effect, impose constraints on the $f(R)$ theories, which facilitates in rejecting non-suitable models of $f(R)$ theories.

The field equation for metric $f(R)$ gravity takes the following form :

$$R_{\mu\nu} = \frac{1}{f_R}\left[T^m_{\mu\nu} + \frac{1}{2}g_{\mu\nu}f - D_{\mu\nu}\right], \tag{3.2}$$

where $f_R \equiv \frac{df}{dR}$, $T^m_{\mu\nu}$ is the matter energy-momentum tensor with the superscript "m" denoting "matter", $g_{\mu\nu}$ is the spacetime metric, and $D_{\mu\nu} = (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\, f_R$ denote the higher-order terms arising out of the spacetime curvature, which act as the source terms for the dark energy.

## 3.2 $f(R,T)$ Theories

Another extension to the $f(R)$ theory of gravity is possible by coupling the trace $T$ of the matter energy-momentum tensor, along with the Ricci scalar $R$ in the Einstein-Hilbert action, and thus replace the function $f(R)$ by $f(R,T)$, which leads to another class of modified theories of gravity, known as the $f(R,T)$ theories. The formalism for $f(R,T)$ gravity was first presented by Harko [9]. The inclusion of the trace $T$ may arise from the existence of imperfect exotic matter. By exotic matter, we mean that these types of matter may have properties which are peculiar to the properties of matter we encounter in real life, for example, negative pressure, or negative energy density. The $f(R,T)$ theories are found to be able to explain the galactic rotation curves, the dark source terms and the late-time acceleration of the universe, and pass the Solar system tests, which, once again as in the case of $f(R)$ theories, impose constraints on the viability of $f(R,T)$ models. However, this problem can be circumvented by making suitable choice of the $f(R,T)$ function in the action of this theory.

The field equation for the metric $f(R,T)$ formalism, is expressed in the following form :

$$R_{\mu\nu} = \frac{1}{f_R}\left[(1+f_T)\,T^m_{\mu\nu} - \mathcal{L}_m g_{\mu\nu} f_T + \frac{1}{2}g_{\mu\nu}f - D_{\mu\nu}\right], \tag{3.3}$$

where $g_{\mu\nu}$ is the metric tensor representing the four-dimensional spacetime in the region interior to the collapsing matter, $R$ is the Ricci scalar, $T$ is the trace of the energy-momentum tensor, $f_R$ and $f_T$ are the derivatives of the $f(R,T)$ function with respect to $R$ and $T$, respectively, $L_m$ is the interior matter Lagrangian, and $D_{\mu\nu} = (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R$ which includes the higher order curvature terms, which acts as the source of dark energy.

In contrast to General Relativity in which curved geometry of spacetime arose from the distribution of matter and energy, that was aesthetically represented in the Einstein's equations which indicated that 'matter curves geometry and the curved geometry in turn influences the motion of test particles', the $f(R,T)$ action includes the Trace $T$ of the matter Lagrangian woven together with the curvature term $R$. As a result, both side of the $f(R,T)$ field equations includes the matter term, unlike GR, and hence the beauty of GR is lost in the $f(R,T)$ theory, although it has some advantage over GR, as explained above.

### 3.2.1 A Note on Minimal and Non-Minimal Coupling

The word "coupling" means a coupling between matter and geometry. In GR, the beauty lies in the fact that the essence of the theory can represented as geometry $\equiv$ matter. In modified gravity, this inter-relation between matter and geometry is still present, although it is modified due to the presence of extra terms which may arise due to higher order curvature terms, or due to the inclusion of matter contribution in the gravitational Lagrangian. Minimal coupling in $f(R,T)$ theory represents a choice of the $f(R,T)$ function in which there are no terms combining both $R$ and $T$, and can be expressed in the form $f_1(R) + f_2(T)$. Non-minimal coupling on the other hand, includes terms which have both $R$ and $T$ dependence, for example $f_1(R) + f_2(R)f_3(T)$ [9]. It is worth mentioning that in modified gravity theories such as $f(R)$ and $f(R,T)$ theories, the covariant derivative of the matter energy-momentum tensor does not vanish, as can be worked out from the action and the field equations. In other words, the energy-momentum tensor is not conserved, leading to an extra force term in the geodesic equation, which describes the motion of a test particle in the gravitational field, and which, in case of General Relativity, does not contain the extra force. Hence unlike in GR, the motion of test particles become non-geodesic in modified gravity theories.

## 3.3 Other Theories of Gravity

Some other modified theories of gravity include $f(R,\mathcal{L}_m)$ theories [11], $f(R,T,R_{\mu\nu}T^{\mu\nu})$ theories [12, 13], and $f(R,T^\phi)$ theories, where $\phi$ is a scalar field, and a host of other theories. Each of these theories involve varying degrees of complexity in the mathematical formulation. In fact, each specific expression of the function $f$ represents one particular theory. So, in principle, we have the entire collection of such theories under the umbrella of any chosen general function. There are also several alternative theories of gravity, for example, the Brans-Dicke theory [14], which is a scalar-tensor theory, and the Gauss-Bonnet theory, where the action is quadratic in the Riemann tensor and the Ricci tensor, and is a particular case of Lovelock gravity [15].

# Chapter 4

# Gravitational Collapse

## 4.1 The Phenomenon

The basic phenomenon of gravitational collapse may be organised in the following manner :

- First, the geometry of the collapsing matter is described : it may have spherical symmetry, cylindrical symmetry, or may also be described by plane-symmetry.
- A clear distinction is made between what constitutes the interior region and the exterior region with respect to the collapsing matter. The boundary of the collapsing matter is taken to segregate the two regions.
- The interior region of spacetime, which contains the collapsing matter, is now described by a suitably chosen four-dimensional spacetime metric, which depends on the geometry of the collapsing matter.
- The nature, or the constituents of the collapsing matter, is described by the appropriate energy-momentum tensor, which decides whether the collapsing body is a dust ball, perfect fluid, anisotropic fluid, or has additional dissipative effects such as radial heat flow, free-streaming raditation, and bulk or shear viscosity. Also, the presence of charge, if any, is represented by Maxwell's electromagnetic stress tensor and combined with the usual uncharged energy-momentum tensor.

- When there are no radiative effects, and the collapsing matter does not give away mass-energy with time, the exterior region may be described as "empty", and described by the Schwarzschild spacetime. When there are radiative effects, however, Vaidya and generalized Vaidya spacetimes have to be used to describe the exterior. The appropriate energy-momentum tensor has to be used for the exterior region as well.
- Finally, the boundary of the collapsing matter is described by a timelike 3D hypersurface.
- The next step is to examine the continuity of certain quantities across the boundary, which are known as junction conditions. Depending upon the theory of gravity chosen, the junction conditions will need additional continuity requirements.
- Considering the collapse velocity to be negative, variation of the mass-energy content of the collapsing body with respect to spatial and temporal coordinates, can be determined.
- Aspects like dynamical stability/instability conditions, formation of apparent horizon, nature of the end state of collapse, and the relation between structure scalars and matter variables can be investigated following this stage.

## 4.2 Gravitational Collapse in GR

The first ever work on gravitational collapse was carried out by Oppenheimer and Snyder in 1939 [16], and by Datt [17], independently. Over the years several groups of researchers have worked on different aspects of gravitational collapse in GR [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35], in presence/absence of pressure anisotropy, radial heat flow, free-streaming radiation, shearing effects, and charge. Each of these factors affect the nature of the collapse in a significant manner.

## 4.3 Gravitational Collapse in $f(R)$ and $f(R,T)$ Gravity

Gravitational collapse was investigated in $f(R)$ gravity as well [36, 37, 38, 39, 40, 41, 42, 43, 44]. Several authors studied gravitational collapse in $f(R,T)$ gravity too

[208, 46, 47, 48, 49, 50, 51, 52, 53] which encompass dynamical conditions, apparent horizon formations and causes of irregularity in energy densities. This list of works carried out in this field is by no means, exhaustive. The shift from General Relativity to modified theories of gravity can significantly affect the progression of the collapse and the nature of the final state of the collapse, because of the higher order curvature terms arising out of the modified gravitational Lagrangian. Different combinations of terms in the energy-momentum tensors have provided different results in the nature of the collapse phenomenon. Collapse of a matter configuration implies an increase in density, and consequently, a variation from the density in other regions of the spacetime. This may also provide insights regarding the formation of clusters and how they might form in the modified gravity spectacle. The next chapter provides an outline of the various aspects of collapse which have been investigated in this thesis.

# Chapter 5

# Important Features of Gravitational Collapse

The various features that characterize the progress of gravitational collapse of a ball of fluid, are the conditions of dynamical stability or instability, formation of horizon (more appropriately, apparent horizon and sometimes the event horizon and/or dynamical horizon), and the nature of the final state of collapse. The relation between structure scalars and matter variables can also be investigated to gather detailed information about the dynamics of collapse.

## 5.1 Dynamical Conditions

In this thesis, we have studied the dynamical conditions for a generalised fluid with spherical symmetry in the absence of charge, in $f(R,T)$ theory of gravity [54]. The interior spacetime is described by the metric

$$ds_-^2 = -A^2dt^2 + B^2dr^2 + C^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.1}$$

where $A, B$ and $C$ are in general functions of both $t$ and $r$. Dynamical conditions for a collapsing matter had been previously studied in GR, $f(R)$, and $f(R,T)$ gravity for various matter configurations [208, 46, 47, 48]. We have considered the generalised matter fluid to involve dissipation in the form of heat flux, radiation, and shear viscosity, as described by the energy-momentum tensor :

$$T^{m}_{\mu\nu} = (\rho + p_{\perp})V_{\mu}V_{\nu} + p_{\perp}g_{\mu\nu} + (p_r - p_{\perp})\chi_{\mu}\chi_{\nu} + q_{\mu}V_{\nu} + V_{\mu}q_{\nu} + \epsilon l_{\mu}l_{\nu} - 2\eta\sigma_{\mu\nu}. \quad (5.2)$$

An $f(R,T)$ function of the form $f(R,T) = R + \alpha R^2 + \lambda T$ has been considered. There are no cross-terms involving both $R$ and $T$, or, in other words, the Ricci scalar and the Trace are minimally coupled. The $f(R,T)$ field equations have been derived after equating the Einstein tensor to an "effective" energy-momentum tensor, which involves terms arising from the higher order Riemann curvature. After taking the covariant derivative of this effective energy-momentum tensor, and projecting it in the direction of the four-velocity vector, two equations have been obtained, one of which describes the temporal variation of the effective energy-momentum tensor, and the other describing the radial variation of the same. These equations, describing the dynamics of the collapsing matter, are known as the dynamical equations. We have adopted a perturbation scheme as previously used by Herrera, Santos, Chan and other collaborators [23, 24, 25, 26]. After finding the static and perturbed configurations of the field equations and the dynamical equations, the Harrison-Wheeler equation of state has been invoked, which is given by

$$\bar{p}_r = \frac{\Gamma p_{r0}}{\rho_0 + p_{r_0}}\bar{\rho}. \quad (5.3)$$

Finally we have obtained our collapse equation which is a non-homogeneous second order differential equation in $D(t)$, the temporal dependence of the perturbation. The Newtonian and Post-Newtonian approximations have been done, to determine a bound for the specific heat ratio $\Gamma$. It was seen that the upper bound for $\Gamma$ reduces in presence of shear, heat flux and radiation, in both the cases. In addition, the instability range in the post-Newtonian approximation is found to be greater than that in the Newtonian approximation. Also, if the dissipative and shearing effects were absent, the collapse equation would have become a homogeneous differential equation, whose solution would have been a negative exponential in the time $t$.

## 5.2 Causal Heat Transport

The casual heat transport equation from Muller-Israel-Stewart theory was then obtained for the collapsing matter, which is given by

$$\tau h^{ab}u^{c}q_{b;c} + q^{a} = -Kh^{ab}(T_{1,b} + T_{1}a_{b}) - \frac{1}{2}KT_1^2\left(\frac{\tau u^b}{KT_1^2}\right)_{;b} q^{a}. \quad (5.4)$$

From this, the expression for the time-derivative of heat flux was extracted and substituted in the second dynamical equation, which represents the radial variation of the effective energy-momentum tensor. This dynamical equation was rearranged in the form of a force equation, with the left-hand side expressed as "mass times acceleration", where the acceleration resulted from the temporal variation of the collapse velocity, and the right-hand side showing the various forces, including the gravitational force, and the hydrodynamic force which involves terms describing the pressure gradient. It was seen that due to the effect of thermal conduction, the inertial mass and the gravitational mass, both decreased by the same factor, thus keeping their ratios unchanged, which is in agreement with the weak equivalence principle. Depending upon whether the value of this factor is negative or positive, the active inertial mass gets reduced or increased accordingly, and the system undergoes expansion or collapse. When the value of this factor is zero, it represents the critical point where the system undergoes a shift between expansion and collapse. At this point, the passive gravitational mass does not affect the collapsing process.

## 5.3 Junction Conditions

The nature of the final state of the gravitational collapse is an area of investigation worth looking into. We have studied the formation of singularity and the apparent horizon for the collapse of a matter configuration with spherical symmetry. Apparent horizon formation was previously studied for charged and uncharged perfect fluid configurations in $f(R)$ gravity and $f(R,T)$ gravity [43, 36, 51, 53]. The chosen matter configuration has been considered to involve isotropic pressure and heat flux, in the absence of charge, free-streaming radiation, and shearing forces, and is given by,

$$T^{m}_{\mu\nu} = (\rho_{int} + p)V_\mu V_\nu + p g_{\mu\nu} + q_\mu V_\nu + V_\mu q_\nu. \tag{5.5}$$

An $f(R,T)$ function of the form $f(R,T) = R + 2\lambda T$ has been considered [198], and, once again, like our previous work [54], there is minimal coupling between the matter and geometry. The region outside the boundary of the collapsing matter has been considered to be described by the generalized Vaidya metric of outgoing radiation, and is given by

$$ds^2_+ = -\left(1 - \frac{2M(v,Y)}{Y}\right) dv^2 - 2dvdY + Y^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{5.6}$$

The energy-momentum tensor in the exterior region is described by a combination of Type-I and Type-II fluids, and is described as

$$T^{+}_{\mu\nu} = \mu l_\mu l_\nu + (\rho + P)\left(l_\mu n_\nu + l_\nu n_\mu\right) + P g^{+}_{\mu\nu}. \tag{5.7}$$

The former represents null-like matter or radiation photons, while the latter represents a perfect fluid component which can be timelike or massive particles. For a type-I fluid, the energy-momentum tensor has only one timelike eigenvector, while there is a double null eigenvector for the energy-momentum tensor corresponding to a type-II fluid. The interior spacetime is considered to be of the most general spherically symmetric form as described in (5.1). The boundary of the collapsing matter, which separates the interior and the exterior spacetimes, is a timelike 3-D hypersurface described by

$$ds^2_\Sigma = -d\tau^2 + \mathcal{R}(\tau)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{5.8}$$

We have applied the $f(R,T)$ junction conditions, first formulated by Rosa [230], which requires, for a smooth matching with no thin shell, the continuity of the spacetime metric, the trace part and trace-free parts of the extrinsic curvature tensor, the Ricci scalar and its first derivative, the trace of the energy-momentum tensor, and its first derivative, across the boundary of the collapsing matter. The Darmois-Israel junction conditions in GR requires only the continuity conditions for the spacetime metric and the extrinsic curvature tensor. It is seen that the matter-Lagrangians for the interior and the exterior spacetimes are required to match at the boundary of the matter, although this is not considered a new result since this initial supposition was used by Rosa in deriving the junction conditions for $f(R,T)$ gravity.

## 5.4 Singularity and Apparent Horizon

Considering the matter Lagrangian for the interior spacetime to be the negative of the interior energy density, the field equations in $f(R,T)$ gravity for our chosen matter configuration have been worked out, and, from the isotropy of pressure, it is found that the interior metric coefficients are interrelated through a 2nd-order differential equation. In order to make the differential equation tractable, the interior metric coefficients have been assumed to be separable in the spatial and temporal coordinates, following Chan [27, 28], and then by a suitable rescaling of the time coordinate, condensing the temporal dependence into the physical radius of the star alone. The resulting form is a second order differential equation in the variable $w(t)$

which describes the temporal dependence of the physical radius. There is a term $D(r)$ in this recasted differential equation, which involves all the terms with spatial, more particularly, radial dependence. The value of this term being zero or non-zero, decides whether the differential equation will be homogeneous or non-homogeneous. The general solution of the differential equation has been obtained with the help of MAPLE-18 software and GRTensor package in both of these cases with unspecified integration constants, the solutions being explicit functional dependences of $w$ on $t$. As the collapse reaches its end, the physical radius of the star shrinks to zero magnitude, which makes $w(t)$ vanish. This is different from the regular centre of the star where the radial coordinate is zero. From vanishing of $w(t)$, the expression for the time of formation of singularity is obtained, with constraints for this time to have a real and positive value. The apparent horizon is a boundary between outward directed light rays which bend inwards, and those which move outwards [198]. The condition for formation of apparent horizon is given by

$$g^{\mu\nu}C_{,\mu}C_{,\nu} = 0. \tag{5.9}$$

From the condition for the formation of apparent horizon, the time of formation of apparent horizon is determined. It is seen that the apparent horizon condition naturally dictates the second-order differential equation to be non-homogeneous. Hence the general solution for that particular case is used hereafter. The difference between the time of formation of apparent horizon and the time of formation of singularity has to be negative, in order for a black hole to be formed as the final state of the collapse. After examining the equations describing this difference, for various cases, the constraints on the unspecified integration constants for the general solution for the non-homogeneous differential equation, have been determined using the standard rules of inequality in algebra. A third constant $\delta$ has come into play, from the condition of apparent horizon, which is shown to be non-zero in order for the collapse to be possible. Choices of this constant being positive or negative, are also found to influence the constraints on the integration constants required for the formation of a black hole. The cases where black hole formation is not possible, are also enlisted in appropriate tables. It is worth mentioning that the presence of shear would have involved more terms involving $w(t)$ and its derivative, in the differential equation obtained from the pressure isotropy condition. This would have made the differential equation more complicated to solve. Also, inclusion of the anisotropy of pressure, in stead of isotropic pressure, would have meant that this method of relating the metric coefficients would not have been possible. In that case, we would have had to use the field equation corresponding to the heat flux, in order to solve for the temporal function $w$, which would have been more complicated. It might also be pointed out that the minimal coupling parameter $\lambda$ , appearing in the $f(R,T)$ function, does not affect the formation of singularity, since it gets eliminated as we utilise the pressure isotropy condition to find the interrelation between the metric coefficients. It also does not affect the formation of the apparent horizon, since the apparent horizon condition remains the same in $f(R,T)$ gravity, as it is in Einstein gravity. It has been shown both in GR and $f(R,T)$ that the presence of shear will slow down the formation of apparent horizon and have a retarding effect on the collapse, and the absence of shear must necessarily result in black hole formation as the final state of the collapse. Hence the constraints on the parameters for black hole formation, would have to be adhered to. Possibilities of naked singularities will arise outside the bounds of these constraints. Further, choosing an $f(R,T)$ function involving higher powers of the Ricci scalar, would have meant that the contributions from the dark source terms would be present in the differential equation, which would have been more difficult to handle. Using the expression for the time variation of the mass-energy content provided in [54], and integrating it from zero to the time of formation of the apparent horizon, an idea about the mass of the black hole formed may be possibly obtained. Also, the black hole will be spherical, since we are considering spherical collapse.

## 5.5 Structure Scalars and Complexity Conditions

The behaviour of different physical parameters of a collapsing matter configuration can be described by a set of scalar quantities known as "structure scalars". To arrive at these scalar quantities, the Riemann curvature tensor is used as the starting point. Bel [56] defined the left, right and double duals of the Riemann tensor. For this, the four-index Levi-Civita symbol was used, which was referred to as the volume element tensor by Bel. The right, left and double duals are expressed as :

$$R^*_{\alpha\beta\gamma\delta} = \frac{1}{2}\eta_{\epsilon\rho\gamma\delta}R^{\epsilon\rho}{}_{\alpha\beta} \tag{5.10}$$

$$^*R_{\alpha\beta\gamma\delta} = \frac{1}{2}\eta_{\alpha\beta\epsilon\rho}R^{\epsilon\rho}{}_{\gamma\delta} \tag{5.11}$$

$$^*R^*_{\alpha\beta\gamma\delta} = \frac{1}{2}\eta_{\alpha\beta}{}^{\epsilon\rho}R^*_{\epsilon\rho\gamma\delta} \tag{5.12}$$

A prescription for the orthogonal splitting of the Riemann tensor was provided in [64]. Following the orthogonal splitting of the Riemann curvature tensor, by taking double projection of each of the three dual tensors derived from it, along the direction of the four-velocity vector, three rank-two tensors were formulated, whose trace parts and trace-free parts came to be called structure scalars. It was Herrera who used the term "structure scalars" for the first time in [58]. These scalars are found to influence the various physical properties of the collapsing matter, such as, the matter-energy density, pressure anisotropy, heat flux, and the active gravitational mass, and different combinations of these structure scalars are sufficient to express the solutions to the Einstein Field Equations in the static case. We have investigated the role of these structure scalars for dissipative collapsing fluid with spherical symmetry in presence of an electromagnetic field in $f(R,T)$ gravity [202]. The contribution of the electromagnetic field is worth looking into. For this particular problem, we have maintained the form of the $f(R,T)$ function to be unspecified, so as to provide a generalized treatment of the entire problem. Following Herrera, we have used the definitions of the left, right and double duals of the Riemann curvature tensor, and subsequently, the rank-two tensors, $Y_{\alpha\beta}$, $X_{\alpha\beta}$ and $Z_{\alpha\beta}$ after an orthogonal splitting of the Riemann tensor, which are given by

$$Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta} u^\gamma u^\delta, \tag{5.13}$$

$$X_{\alpha\beta} = ^* R^*_{\alpha\gamma\beta\delta} u^\gamma u^\delta = \frac{1}{2}\eta^{\epsilon\rho}{}_{\alpha\gamma} R^*_{\epsilon\rho\beta\delta} u^\gamma u^\delta, \tag{5.14}$$

and,

$$Z_{\alpha\beta} = ^* R_{\alpha\gamma\beta\delta} u^\gamma u^\delta = \frac{1}{2}\eta_{\alpha\gamma\epsilon\rho} R^{\epsilon\rho}{}_{\beta\delta} u^\gamma u^\delta. \tag{5.15}$$

The expressions for the trace part and trace-free parts of these rank-two tensors have been specified, and are called structure scalars. These structure scalars are five in number, for the spherically symmetric case, and are denoted by $X_T$, $X_{TF}$, $Y_T$, $Y_{TF}$, and $Z$, and may be expressed via the equations below.

$$X_{\alpha\beta} = \frac{1}{3} X_T h_{\alpha\beta} + X_{TF}\left(\chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha\beta}\right), \tag{5.16}$$

$$Y_{\alpha\beta} = \frac{1}{3} Y_T h_{\alpha\beta} + Y_{TF}\left(\chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha\beta}\right), \tag{5.17}$$

and

$$Z = \sqrt{Z_{\alpha\beta} Z^{\alpha\beta}}. \tag{5.18}$$

Any physical property of a self-gravitating system, other than isotropic pressure and homogeneous energy density, gives rise to complexity within the system. For example, inhomogeneous energy density, anisotropy of pressure, effects of dissipation and shear, electromagnetic field, and all other parameters representing departure from the simplest configuration of a fluid (having homogeneous energy density and pressure isotropy), all add to the complexity of the collapsing fluid. Conventionally, the simplest configuration is said to possess a vanishing complexity. This new definition of complexity was introduced by Herrera [222] for the case of static spherical gravitational collapse, based on a quantity that arises from the orthogonal splitting of the Riemann tensor, in the context of general relativity. The objective was to provide an observable measure of a basic property representing the structures existing within the system, which may not necessarily be a physical structure of the system. Logically, the definition of complexity must include not only a measure of the order within the system or the amount of information available about the system, but also other relevant factors which may be considered suitable for such a characterization of a collapsing fluid. Therefore, for a self-gravitating system, the basic assumption was that the simplest system is represented by a fluid with isotropic pressure and homogeneous in its energy density. Furthermore, to make the definition of complexity meaningful, it should be related to generic properties of the structure of the fluid.

In our work, we have chosen to describe the spacetime interior to the collapsing fireball with the most general spherically symmetric metric. Utilising this particular form of the line element, we have expressed the structure scalars in terms of the metric coefficients and have discussed the effect of complexity on the progress of collapse for the collapsing matter under our consideration.

We have chosen our matter configuration to involve energy density, pressure anisotropy, dissipation in the form of radial heat flow and free streaming radiation, and shear viscosity. The contribution of the charge is brought in by the electromagnetic stress-energy tensor. In the absence of magnetic field, the electromagnetic four-potential will have the scalar potential $\varphi$ as the only non-vanishing component. The components of the vector potential will vanish if there is no magnetic field. The Maxwell field equations, and the vanishing of the covariant derivative of the four-current, which is a condition required by the conservation of charge, enable us to obtain an expression for the total charge $Q$. Since we have chosen a co-moving coordinate system, the charge is assumed to be at rest with respect to this coordinate system, and hence, no vector potential or, accordingly magnetic field comes into play. Hence the total charge is independent of time, although this would not hold in a non-comoving coordinate system. The expressions for the shear scalar, the expansion scalar, and the mass-function, which is the total energy contained inside the boundary of the collapsing body, are then provided in terms of the metric coefficients. The electric part and the magnetic part of the Weyl tensor are provided, where the latter has all its components vanishing in the spherically symmetric case. Expressing the electric part of the Weyl tensor in terms of the Weyl scalar, the non-zero components of the Weyl tensor are enlisted, expressed in terms of the mass-function and the shear scalar. The $f(R, T)$ field equations with the presence of charge are now obtained, and the mass function and the structure scalars are expressed in terms of the matter variables. It is seen that the contribution of the electric charge $Q$ causes an increment in the mass-energy content of the collapsing sphere, as described by the mass function, and also in the structure scalars $X_T$ and $Y_{TF}$. A decrement, however, is caused in the structure scalars $Y_T$ and $X_{TF}$. The matter variables are clubbed together appropriately to represent the "effective energy density", "effective radial pressure", "effective tangential pressure" and the "effective pressure anisotropy".

Utilising the Raychaudhuri equation [59], and the expressions for our structure scalars written in terms of the physical matter variables, we find that the evolution of the shear scalar is controlled by the structure scalar $Y_{TF}$, and the evolution of the expansion scalar is controlled by $Y_T$. The evolution of the shear and expansion scalars involve the covariant derivatives of the respective scalars. The relation between $X_{TF}$ and the inhomogeneity of the effective energy density, and their relation to $Z$ is also brought out clearly. The contribution from the electric charge has been deliberately not absorbed into the effective variables, so as to highlight the contribution arising due to the presence of charge.

It is seen from the analysis, that when there is no heat dissipation, or when the coefficient of the radial heat flow vanishes, the inhomogeneity in the energy density is influenced by the structure scalar $X_{TF}$, and the mass function. Further, the presence of charge also affects the effective energy density inhomogeneity and $X_{TF}$. The heat dissipation and the inhomogeneity of the energy density is also controlled by $Z$, in addition to $X_{TF}$. The evolution of the expansion scalar is controlled by $Y_T$, while the expansion of the shear scalar is controlled by $Y_{TF}$. While the contribution of charge does not appear explicitly in these two evolution equations, it must be kept in mind that the expressions for the structure scalars $Y_T$ and $Y_{TF}$ already have the contribution of the electric charge manifested in them. The effective homogeneous energy density is controlled by the structure scalar $X_T$, while the anisotropy of pressure is influenced by both $Y_T$ and $Y_{TF}$. The electric charge $Q$ is found to appear in the expressions for all the four structure scalars $X_T$, $X_{TF}$, $Y_T$ and $Y_{TF}$, causing an increment in $X_T$ and $Y_{TF}$, and a decrement in $Y_T$ and $X_{TF}$. Also, the structure scalar $Z$ can be expressed in terms of the mass function, the total charge and $X_{TF}$. The presence of charge also causes an increase in the mass-energy content of the collapsing fireball. We have also discussed a special case, which examines the relations between the matter variables and the structure scalars for a constant $R$ and $T$ condition for a dust ball. In case of a dust ball, except the matter-energy density, all the other physical variables, viz. the radial and tangential pressure components, heat flux, free-streaming photon radiation, shear, and the charge will vanish, and the effective energy density will be identical to the matter-energy density. In this particular case, the sum of the structure scalars $\tilde{X}_{TF}$ and $\tilde{Y}_{TF}$ becomes zero, where the tilde denotes constant $R$ and $T$ conditions. Also individually, their magnitudes are the same as that of the Weyl scalar, only with a difference in sign. We have also assumed a simple $f(R,T)$ function, linear in both $R$ and $T$, and expressed the relations between the structure scalars and the matter variables. We have also shown that when all dissipative and electromagnetic effects disappear, i.e., for a collapsing matter with isotropic pressure and homogeneous energy density, the structure scalar $Y_{TF}$ vanishes. Hence this particular structure scalar have been regarded as the complexity factor of the collapsing matter.

All these aspects have been separately discussed in the publications presented in the following chapters of this thesis. The equations provided in this chapter, will be developed further, later in the thesis.

# Part II

# Some Aspects of Gravitational Collapse in $f(R, T)$ Gravity

In this part of the thesis we report the investigations that have been carried out in course of this Ph.D. program. The results of this research work is contained in three separate chapters, which deal with the work that have been already published in peer-reviewed international journals.

# Chapter 6

# Dynamical Stability and Causal Transport

This chapter discusses the bounds on the adiabatic index for dynamical stability, as well as the causal transport phenomenon for the spherical collapse of a dissipative matter configuration in $f(R,T)$ gravity. The publication details of the material in this chapter are provided below.

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## 6.1 Introduction

The final outcome of stellar evolution depends not only on the size of the object undergoing collapse [60] but also on other physical parameters [61]. On account of the high energy dissipation during collapse, more massive stars tend to be more unstable because of massive radiation transport and rapid loss of nuclear fuel [61, 62]. The effects of dissipation and slight change in isotropy alter the subsequent evolution of the collapsing system considerably. Several researchers [18, 23, 28, 29, 64, 24, 25, 26, 19, 63, 20] have studied extensively the phenomenon of gravitational collapse of fluids in presence or absence of various conditions like anisotropy, radiation, dissipation, expansion, shear, along with the conditions of dynamical instability and causal transport phenomenon occurring during gravitational collapse, and the list is even longer. The subject of spherical gravitational collapse has always been at the heart of astrophysical investigations on account of the symmetry of the matter distribution, leading to simplification of the field equations, and also due to the fact that some classes of realistic gravitational collapse can be modeled as spherical collapse with only small deviations. The analysis is also simplified due to the absence of gravitational waves. Studies of radiating spherical collapse in the presence of heat flow [65, 66], relaxational effects in stellar heat transport during collapse [67] and temperature profile of such a collapse within the framework of causal thermodynamics have been conducted [68]. Some authors [35, 31] have also studied the dissipative collapse of charged configurations.

The limitations of General Relativity (GR) on large scales led to the investigation of astrophysical processes in modified theories of gravity, e.g., theories which provide improved models of the accelerating universe. Among these modified theories (some of the review articles are listed in [69, 70, 8, 71, 72, 73, 74, 75]), the $f(R)$ gravity presents a very elementary modification by including higher order curvature terms to incorporate the dark energy components, as well as the inflationary phase [76, 77, 78, 79, 80, 81, 82, 83]. Not only does $f(R)$ gravity reproduce the $\Lambda$CDM epoch, or is able to mimic the cosmological constant at the present epoch, but it can also unify the entire evolution history of the universe [84, 85, 86, 99, 109]. The null dust non-static exact solutions in $f(R)$ gravity constrained by constant curvature describing anti de-Sitter background evolution, was studied by Ghosh and Maharaj [87]. Cembranos et al. [88] studied the evolution of gravitating sources in the presence of dust fluid in a general $f(R)$ model with a view to determine the possible constraints on such models. Goswami et. al. studied the collapse of massive stars in $f(R)$ gravity and showed that the extra matching conditions arising in modified gravity impose strong constraints on the stellar structure and thermodynamic properties [89]. Chakrabarti and Banerjee investigated the collapse of a perfect fluid source described by Lemaitre-Tolman-Bondi type geometry [43]. Sharif and Yousaf studied the dynamical instability of charged spherical collapse in expansion-free condition [38], and the stability of dissipative charged spherical collapse in the CDTT-$f(R)$ model [40].

In 2011, Harko et al. [9] introduced the $f(R,T)$ theory of gravity, based on the non-minimal matter-to-geometry coupling considered by earlier workers [11, 90, 91]. The action in $f(R,T)$ gravity includes an arbitrary function of the Ricci scalar $R$ and the trace $T$ of the energy-momentum tensor. The inclusion of $T$ takes care of quantum effects or the existence of exotic matter. A combination of a term quadratic in $R$ (as in the Starobinsky model [92]) along with linear terms in $R$ and $T$ as used in [93], provides a useful $f(R,T)$ function. It was realized [52] that a strong coupling of the curvature $R$ with the trace $T$ leads to the violation of the conventional equation of continuity, as a result of which, an extra force arises in the geodesic equation and test particles do not follow a geodesic motion. However, this problem can be resolved by choosing a suitable $f(R,T)$ function. These features of $f(R,T)$ gravity make it *a candidate suitable for* the investigation of various astrophysical phenomena within the context of this theory. The astrophysical, cosmological and thermodynamic implications of $f(R,T)$ gravity have been widely studied by several groups [94, 95, 96, 97, 98, 100]. Houndjo [101] demonstrated a possible unification of the accelerated expansion phase with the matter dominated era in cosmic evolution by choosing a viable $f(R,T)$ function subject to appropriate constraints, which ensure that either no ghost state arises where dark energy becomes responsible for the accelerated expansion, or no tachyon arises. Subsequently, Odintsov and Sáez-Gómez [13] proposed the $f(R,T,R_{\mu\nu}T^{\mu\nu})$ theory as an extension of $f(R,T)$ gravity. They derived the general FRLW field equations in the presence of the $R_{\mu\nu}T^{\mu\nu}$ coupling terms and studied several cosmological solutions, thereby establishing that thematter sector behaves differently from that in GR, because the continuity equations are different. In this theory too, the problem can be dealt with by considering an appropriate $f(R,T,R_{\mu\nu}T^{\mu\nu})$ function, so that one can assume the usual evolution for a perfect fluid matter even in the presence of the extra degree of freedom.

Sharif and Yousaf [208] considered the dynamical analysis in $f(R,T)$ gravity with non-null expansion scalar at Newtonian and post-Newtonian eras and examined the role played by matter variables on stellar structure. Others investigated the dynamical instability of spherical stars in locally anisotropic environment in $f(R,T)$ theory under various situations like expansion-free, shear-free conditions and from other perspectives [46, 47, 48]. Yousaf et al. [50] studied the factors leading to irregularities in the case of a spherically symmetric self-gravitating star consisting of dissipative matter and radiation in $f(R,T)$ gravity. In another work [102], they discussed the advantages of working with modified gravity. If the curvature is low, one can observe accelerated expansion of the universe, while for high value of curvature, the singularities can be made smoother. They [103] also studied the effect of charge on spherically symmetric gravitational vacuum stars (gravastars) in

$f(R,T)$ gravity. Yousaf [228] considered a charged cylindrically symmetric gravastar with perfect fluid matter content in $f(R,T)$ gravity. It was found that for an increase in the amount of charge in the gravastar, the proper length of its middle thin shell reduces. Also, the energy content in the gravastar reduces for a rise in the amount of electric charge. The entropy of the system increases if the charge decreases or with an increase in the thickness of the shell. Bhatti et al. [105] considered a spherically symmetric model of a gravastar and studied some of its characteristics in $f(R,G)$ theory, where $G$ is the Gauss-Bonnet term.

The physical behavior, stability and validity of energy conditions of compact objects under the effect of charge was examined in [106], and the collapsing and expanding solutions of anisotropic charged cylinder was determined in [107]. Some authors have studied the higher-dimensional perfect fluid collapse in $f(R,T)$ theory [108]. The effects of charge on the stability range of anisotropic spherical stellar model in the framework of $f(R,T)$ gravity have been studied in [52]. In the paper [110] the exact solutions for non-static anisotropic self-gravitating source in $f(R,T)$ were derived, and the nature of singularity and trapped surfaces during the collapse of a charged perfect fluid was determined in [53]. They also studied the thermodynamic aspects of a viscous dissipative collapse in $f(R,T)$ gravity [111]. Recently the complexity factor of a self-gravitating system was determined through the orthogonal splitting of the Riemann tensor and the behaviour of the complexity for a cylindrically symmetric system in $f(R,T)$ theory of gravity was studied [112].

A spherically symmetric perfect fluid model for a gravastar in $f(R,T,Q)$ theory wasstudied in [113], where $Q = R_{\mu\nu}T^{\mu\nu}$. The paper [114] studied the instability criteria by the method of dynamical analysis for a cylindrically symmetric, anisotropic radiating matter fluid in $f(R,T,Q)$ theory. The linear perturbation scheme was invoked under the assumption that the fluid is initially in hydrostatic equilibrium, with dependence only on the radial coordinate, but later it begins to undergo radial oscillations. The instability ranges were calculated in the Newtonian and post-Newtonian regimes, utilizing the Harrison-Wheeler equation of state. The same authors [115] discussed the instability ranges of compact stars by the method of dynamical analysis in $f(R,T,Q)$ theory and adopting the linear perturbative scheme. The energy-momentum tensor was that of an anisotropic perfect fluid, and the interior metric was the most general spherically symmetric with nonzero expansion scalar. Yousaf, Bamba, Bhatti and Farwa [116] studied the dynamical evolution of a spherical anisotropic fluid with nonvanishing expansion scalar using perturbative scheme and its instability ranges calculated at the Newtonian and post-Newtonian regimes.

Gravitational collapse ensues only when a self-gravitating object is perturbed from its initial hydrostatic equilibrium condition, when the internal pressure cannot counteract the gravitational contraction. This leads to the formation of compact objects like white dwarfs, neutron stars, black holes, etc. The effect of various factors in the formation of these objects can hardly be undermined. In this context, Yousaf [117] explored the behavior of scalar variables associated with the shearing viscous dissipative anisotropic spherical stars in the framework of $f(G,T)$ theory. A correspondence was drawn between the metric scale factors, tidal forces and structure parameters of the collapsing configuration. As the technique of orthogonal breaking of the Riemann tensor helps us to study the reasons behind the emergence of inhomogeneities in the initially regular spheres as the collapse progresses, this technique was used to determine the modified structure scalars, and the role of these invariants in the evolutionary properties of radiating spheres was explored. The analysis showed that the evolutionary phases of the spherical interiors can be well understood in terms of extended versions of scalar variables. The shear evolution equation (SEE), the expansion evolution equation (EEE) - also known as the Raychaudhuri equation, and the Weyl differential equation (WDE) can be expressed with the help of these structure scalars. For constant $G$ and $T$ , in case of a dust ball, the cause of inhomogeneous energy density in the compact object which is initially regular can be expressed through theWDE using one of the structure scalars [118]. Depending on the mass of the collapsing object [119, 120], the collapse can result in formation of structures like black holes, neutron stars or white dwarfs.

It was Misner and Sharp [121] and Misner [122], who provided a complete description of the dynamical equations for adiabatic and dissipative collapse. In fact, all the physical parameters associated with the collapsing matter should obey the transport equations of causal thermodynamics. Earlier Tolman [123] had suggested that the heat energy associated with the collapsing matter should have an inertia associated with it, and this inertial term appears in the transport equations derived by Eckart and Landau-Lifshitz [124, 125]. However the Landau-Eckart prescription could only be solved by a hyperbolic theory involving second-order entropy terms, as done by Muller, Israel and Stewart [126, 127, 128, 129]. The dynamical equation can be recasted into the form of Newton's Second Law of Motion. The transport equation can be expressed in terms of the acceleration of the collapsing system, and after we couple the transport equation with the dynamical equation, both the inertial mass and the gravitational mass are found to get reduced by the same factor, in agreement with the principle of equivalence. The thermal dissipative effects reduce the effective inertial mass term, and the inertial mass appears to lag behind the collapsing matter as the collapse progresses [130].

Herrera and Santos [29] discussed the importance of dissipative processes in gravitational collapse and extended the Misner dynamical equations to include dissipation in the form of radial heat flow along with pure radiation in order to couple the dynamical equations with the transport equations. Herrera et al. [31] applied the same prescription to charged, dissipative, spherically symmetric system with shear viscosity. The paper [133] examined spherically symmetric collapse with heat flow, radiation, shear and bulk viscosity. They applied a full causal treatment to all dissipative variables, and further included the viscous or heat coupling coefficients, which are known to influence the evolution of neutron stars [134]. Plane symmetric gravitational collapse in the Misner and Sharp formalism was presented for a fluid undergoing dissipation in the form of heat flow, radiation, shear and bulk viscosity, and the dynamical equations were coupled with the causal transport equations [135]. The gravitational collapse of cylindrical viscous heat conducting anisotropic fluid was investigated in [136], and the transport equation was derived to determine its effect on the collapsing system. The effect of dissipation on the dynamics of charged bulk viscous collapsing cylindrical source with heat flux was studied in [137]. Hydrodynamic and thermodynamic analysis of gravitational collapse of a locally anisotropic shear-free fluid with dissipative flows was presented in [138]. The exact model satisfied all the energy conditions throughout the interior of the star and for the entire duration of collapse process and provided a physically viable temperature profile in the context of causal thermodynamics.

Spherically symmetric collapse of isotropic non-viscous fluid satisfying the energy conditions together with matter violating the null energy conditions was investigated in the framework of $f(R)$ theory and effect of such matter on the coupling of dynamical and transport equations, was studied in [139]. Dissipative gravitational collapse in non-static cylindrical symmetric geometry was examined in [140] by using Misner-Sharp formalism in the framework of metric $f(R)$ gravity, and the dynamical equation was coupled with the full causal transportation equations in the context of Israel-Stewart formalism. Static configurations and radial stability of compact stars have also been studied within the context of $f(R,T)$ gravity [141]. Recently, the dynamics of non-adiabatic charged spherical gravitational collapse of an anisotropic fluid with heat flux was analyzed in the framework of $f(R,T)$ gravity using the Misner-Sharp approach [142]. However, the authors did not consider the effect of shear and free-streaming radiation, nor were the instability conditions analysed.

In this paper, we intend to explore the dynamical instability and the causal heat transport of an uncharged anisotropic fluid with shear viscosity in the presence of free-streaming radiation, undergoing spherically symmetric collapse in $f(R,T)$ gravity. The manuscript is organized as follows: In Section 6.2, we present the $f(R,T)$ formalism, followed by the choice of the interior metric and the corresponding field equations in Section 6.3. The dynamical equations are laid down in Section 6.4, and the perturbation analysis along with the Newtonian and post-Newtonian approximation is presented in Section 6.5. The transport equations are derived in Section 6.6. Finally, the summary of our findings and the final outlook are presented in Section 6.7.

## 6.2 The $f(R,T)$ formalism

The modified Einstein-Hilbert action in $f(R,T)$ gravity is given by [9]

$$S=\int d^4x\sqrt{-g}\left(\frac{f(R,T)}{16\pi G}+\mathcal{L}_m\right), \tag{6.1}$$

where $g$ is the determinant of the 4D metric $g_{\mu\nu}$, $R$ is the Ricciscalar, $T$ is the trace of the stress-energy tensor $T_{\mu\nu}$, $G$ is the Newton's gravitational constant, $\mathcal{L}_m$ denotes the Lagrangian density for matter fields, and $f(R,T)$ is a suitable analytic function of $R$ and $T$ that describes the gravitational interaction. The corresponding field equations of $f(R,T)$ gravity has the form:

$$\begin{aligned} R_{\mu\nu}f_R(R,T)-\frac{1}{2}g_{\mu\nu}f(R,T)+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_R(R,T)& \\ =8\pi GT^m_{\mu\nu}-f_T(R,T)\left(T^m_{\mu\nu}+\Theta_{\mu\nu}\right)&, \end{aligned} \tag{6.2}$$

where $f_R(R,T)$ and $f_T(R,T)$ denote the derivatives of $f(R,T)$ with respect to $R$ and $T$ respectively, $\Box=g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian operator, $\nabla_\mu$ is the covariant derivative associated with the Levi-Civita connection of the metric tensor, and $\Theta_{\mu\nu}$ is defined by

$$\Theta_{\mu\nu}=g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}=-2T^m_{\mu\nu}+g_{\mu\nu}\mathcal{L}_m-2g^{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}. \tag{6.3}$$

The covariant divergence of eq. (6.2) gives [95]

$$\begin{aligned} \nabla^\mu T^m_{\mu\nu}= \ & \tfrac{f_T(R,T)}{8\pi G-f_T(R,T)}\left[(T^m_{\mu\nu}+\Theta_{\mu\nu})\nabla^\mu\ln f_T(R,T)\right. \\ & \left.+\nabla^\mu\Theta_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}\nabla^\mu T\right]. \end{aligned} \tag{6.4}$$

Thus the conventional energy-momentum tensor is not conserved, unlike in GR, which means that particles do not follow *geodesic paths* in pure gravitational fields in the framework of $f(R,T)$ gravity.

While writing out the field equations, various types of $\mathcal{L}_m$ can be chosen to model different types of matter. For this anisotropic fluid, we choose $\mathcal{L}_m = -\rho$, where $\rho$ is the energy density of the matter distribution, so that (6.3) reduces to the form

$$\Theta_{\mu\nu} = -2T^m_{\mu\nu} - \rho g_{\mu\nu}.$$

Consequently, (6.2) becomes

$$\begin{aligned} R_{\mu\nu} f_R(R,T) - \frac{1}{2} g_{\mu\nu} f(R,T) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R(R,T) \\ = 8\pi G T^m_{\mu\nu} + f_T(R,T)\left(T^m_{\mu\nu} + \rho g_{\mu\nu}\right). \end{aligned} \tag{6.5}$$

Faraoni [96] showed that both $\mathcal{L}_m = p$ and $\mathcal{L}_m = -\rho$, where $p$ is the isotropic pressure of matter, are equivalent when the fluid couples minimally to gravity. However, this freedom of choice is lost when the pressure is not isotropic.

Equation (6.5) can be written in the form of an effective field equation as follows:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_{eff} T^m_{\mu\nu} + T^D_{\mu\nu}, \tag{6.6}$$

where

$$G_{eff} = \frac{1}{f_R(R,T)}\left(G + \frac{f_T(R,T)}{8\pi}\right)$$

is the effective gravitational coupling in $f(R,T)$ gravity, and

$$\begin{aligned} T^D_{\mu\nu} =& \frac{1}{f_R(R,T)}\Big[\rho g_{\mu\nu} f_T(R,T) + \frac{1}{2}(f(R,T) - R f_R(R,T)) g_{\mu\nu} \\ &+ (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box) f_R(R,T)\Big] \end{aligned} \tag{6.7}$$

represents the contribution to the energy-momentum tensor from the matter-curvature coupling and also includes non-equilibrium description of the field equations.

The full form of eq. (6.6) in $f(R,T)$ gravity is then

$$\begin{aligned} G_{\mu\nu} = T^{eff}_{\mu\nu} =& \frac{1}{f_R}\Big[(8\pi G + f_T) T^m_{\mu\nu} + \rho g_{\mu\nu} f_T + \frac{1}{2}(f - R f_R) g_{\mu\nu} \\ &+ (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box) f_R\Big], \end{aligned} \tag{6.8}$$

where $T^{eff}_{\mu\nu}$ is the effective energy-momentum tensor incorporating gravitational effects. Our equation (6.8) follows the convention adopted by several authors beginning with Harko and his collaborators (see for example [9, 90, 94, 95, 208, 46, 47, 48, 52]). However, instead of the term $(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f_R(R,T)$, Yousaf et. al [50] considered the term $(\nabla_\mu\nabla_\nu + g_{\mu\nu}\Box)\, f_R(R,T)$ in the $f(R,T)$ field equations (2) in their paper. This led to a difference in the various components of the field equations obtained by them. Here we have retained the expression given in [9].

In the subsequent analysis we will assume $8\pi G = 1$ for the sake of simplicity of presentation.

## 6.3 Interior metric and Field equations for $f(R,T)$ gravity

The physical system consists of a timelike 3D bounding surface, $\Sigma$, which divides the 4D spacetime into an interior and an exterior portion, denoted by $V^-$ and $V^+$ respectively. We assume that the region inside $\Sigma$ is modeled by the general non-static non-rotating spherically symmetric metric

$$ds^2_- = -A^2dt^2 + B^2dr^2 + C^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6.9}$$

where $A, B$ and $C$ are in general functions of both $t$ and $r$. The interior spacetime is filled with shearing viscous, locally anisotropic and radiating fluid described by the energy-momentum tensor $T^m_{\mu\nu}$:

$$T^m_{\mu\nu} = (\rho + p_\perp)V_\mu V_\nu + p_\perp g_{\mu\nu} + (p_r - p_\perp)\chi_\mu\chi_\nu + q_\mu V_\nu + V_\mu q_\nu + \epsilon l_\mu l_\nu - 2\eta\sigma_{\mu\nu}, \tag{6.10}$$

where $\rho$ is the energy density, $p_\perp$ the tangential pressure, $p_r$ the radial pressure, $q^\mu$ the heat flux, $\epsilon$ is the radiation density, $V^\mu$ is the time-like four velocity of the fluid, $\chi^\mu$ is a unit spatial vector along the radial direction, $l^\mu$ is a null 4-vector, and $\eta$ $(> 0)$ is the coefficient of shearing viscosity, respectively, such that

$$V^\mu V_\mu = -1, \;\; V^\mu q_\mu = 0, \;\; \chi^\mu\chi_\mu = 1, \;\; \chi^\mu V_\mu = 0, \;\; l^\mu V_\mu = -1, \;\; l^\mu l_\mu = 0. \tag{6.11}$$

In comoving coordinates, we have

$$V^\mu = A^{-1}\delta^\mu_0, \quad \chi^\mu = B^{-1}\delta^\mu_1, \quad l^\mu = A^{-1}\delta^\mu_0 + B^{-1}\delta^\mu_1, \quad q^\mu = B^{-1}q\delta^\mu_1 = B^{-1}(0,\, q,\, 0,\, 0), \tag{6.12}$$

where $q = q(r,t)$ and $\epsilon = \epsilon(r,t)$. The acceleration vector $a^\mu$, the expansion scalar $\Theta_1$ and the magnitude of the shear scalar $\sigma$ are given by [28, 143]:

$$a^\mu = \left(0, \frac{A'}{AB^2}, 0, 0\right), \tag{6.13}$$

$$\Theta_1 = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{C}}{C}\right), \tag{6.14}$$

$$\sigma = -\frac{1}{3A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \tag{6.15}$$

where primes and dots on the metric functions denote partial derivatives with respect to $r$ and $t$ respectively.

The components of the Einstein tensor for the interior metric are given by

$$G_{00} = A^2\left[\frac{2}{A^2}\frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{\dot{C}^2}{A^2C^2} + \frac{1}{C^2} - \frac{1}{B^2}\left(\frac{2C''}{C} + \frac{C'^2}{C^2} - \frac{2B'}{B}\frac{C'}{C}\right)\right], \tag{6.16}$$

$$G_{11} = B^2\left[\frac{1}{A^2}\left(-\frac{2\ddot{C}}{C} - \frac{\dot{C}^2}{C^2} + \frac{2\dot{A}}{A}\frac{\dot{C}}{C}\right) + \frac{1}{B^2}\left(\frac{C'^2}{C^2} + \frac{2A'}{A}\frac{C'}{C}\right) - \frac{1}{C^2}\right], \tag{6.17}$$

$$\begin{aligned} G_{22} = C^2\Bigg[&-\frac{1}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} - \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\ddot{C}}{C}\right) \\ &+\frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'}{A}\frac{B'}{B} + \frac{A'}{A}\frac{C'}{C} - \frac{B'}{B}\frac{C'}{C} + \frac{C''}{C}\right)\Bigg], \end{aligned} \tag{6.18}$$

$$G_{01} = 2\left(-\frac{\dot{C}'}{C} + \frac{A'}{A}\frac{\dot{C}}{C} + \frac{\dot{B}}{B}\frac{C'}{C}\right). \tag{6.19}$$

The Ricciscalar for the interior metric (8.11) is given by

$$\begin{aligned} R = &-\frac{2}{A^2}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{2\dot{A}\dot{C}}{AC} - \frac{\ddot{B}}{B} - \frac{2\dot{B}\dot{C}}{BC} - \frac{\dot{C}^2}{C^2} - \frac{2\ddot{C}}{C}\right) \\ &-\frac{2}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB} + \frac{2A'C'}{AC} + \frac{C'^2}{C^2} - \frac{2B'C'}{BC} + \frac{2C''}{C}\right) + \frac{2}{C^2}. \end{aligned} \tag{6.20}$$

We find that the Kretschmann scalar for the interior space-time is given by a lengthy expression and has $\sim 30$ non-zero components [144]. It involves inverse powers of the field functions $A$, $B$ and $C$. Therefore, both the Ricci and the Kretschmann scalars will diverge at the time $t = t_s$, when these functions tend to zero, indicating a possible curvature singularity. However, for an exact analysis we need to find the solution to the $f(R,T)$ field equations in our case, a fairly elaborate program which will be presented elsewhere.

The $f(R,T)$ field equations (6.8) for the spherical non-static interior (8.11) are found as

$$G_{00} = \frac{A^2}{f_R}\left[\rho + \epsilon(1+f_T) - \frac{1}{2}(f - Rf_R) + \frac{\zeta_{00}}{A^2}\right], \tag{6.21}$$

$$G_{01} = \frac{AB}{f_R}\left[-(1+f_T)(q+\epsilon) + \frac{\zeta_{01}}{AB}\right], \tag{6.22}$$

$$G_{11} = \frac{B^2}{f_R}\left[(1+f_T)(p_r + \epsilon + 4\eta\sigma) + \rho f_T + \frac{1}{2}(f - Rf_R) + \frac{\zeta_{11}}{B^2}\right], \tag{6.23}$$

$$G_{22} = \frac{C^2}{f_R}\left[(1+f_T)(p_\perp - 2\eta\sigma) + \rho f_T + \frac{1}{2}(f - Rf_R) + \frac{\zeta_{22}}{C^2}\right], \tag{6.24}$$

where

$$\zeta_{00} = \left[-\left(\frac{\dot{B}}{B} + \frac{2\dot{C}}{C}\right)\dot{f}_R + \frac{A^2}{B^2}\left(\left(\frac{2C'}{C} - \frac{B'}{B}\right)f'_R + f''_R\right)\right], \tag{6.25}$$

$$\zeta_{01} = \left[\dot{f}'_R - \frac{A'}{A}\dot{f}_R - \frac{\dot{B}}{B}f'_R\right], \tag{6.26}$$

$$\zeta_{11} = \left[\frac{B^2}{A^2}\left(\ddot{f}_R + \left(\frac{2\dot{C}}{C} - \frac{\dot{A}}{A}\right)\dot{f}_R\right) - \left(\frac{A'}{A} + \frac{2C'}{C}\right)f'_R\right], \tag{6.27}$$

$$\begin{aligned}\zeta_{22} = &\left[\frac{C^2}{A^2}\left(\ddot{f}_R + \left(-\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\dot{f}_R\right)\right.\\ &\left.+\frac{C^2}{B^2}\left(\left(-\frac{A'}{A} + \frac{B'}{B} - \frac{C'}{C}\right)f'_R - f''_R\right)\right].\end{aligned} \tag{6.28}$$

## 6.4 Dynamical Equations for the collapsing system

In his pioneering work, Chandrasekhar showed within the framework of GR [145], that even under conditions of hydrostatic equilibrium, a self-gravitating gaseous system becomes dynamically unstable (with respect to radial oscillations) much before they reach the 'Schwarzschild limit'. In Newtonian gravity, the condition of dynamical instability is related to the "ratio of specific heats" of the system, and a difference in the nature of the gravitational field — general relativistic or other, decides the exact condition of stability of such a system. In fact, different degrees of instability will affect the nature of evolution of the collapsing object, thereby leading to a difference in the end state of collapse [146]. The problem of stability has therefore been widely investigated, and is done by examining the dynamical equations of collapse.

The dynamical equations describe the evolution of the parameters of the collapsing stellar object with time and radius. These equations are derived from the conservation laws. We consider the conservation of the Einstein tensor because the matter energy momentum tensor has a non-vanishing divergence in $f(R,T)$ gravity, as is evident from (6.4). The divergence of the effective energy-momentum tensor in (6.8) provides the following continuity equations:

$$\begin{aligned}
&\frac{\dot{f}_R}{f_R^2 A}\left[\rho+\epsilon\left(1+f_T\right)-\frac{f}{2}\right]+\frac{\left(q+\epsilon\right)\left(1+f_T\right)}{B f_R}\left[\frac{f_R'}{f_R}-\frac{4 f_T}{\left(-1+f_T\right)}\left(\frac{A'}{A}+\frac{C'}{C}\right)\right]\\
&+\frac{\dot{f}-\dot{R} f_R}{2 A f_R}+\frac{f_T}{f_R A\left(-1+f_T\right)}\left[-\frac{2A}{B}\left(1+f_T\right)\left(\epsilon'+q'\right)-2\left(\dot{\rho}+\frac{f_T' A\left(q+\epsilon\right)}{B}\right)\right.\\
&+\frac{\left(1+f_T\right)\dot{T}}{2}-2\dot{f}_T\epsilon-2\left(1+f_T\right)\dot{\epsilon}-2\left(1+f_T\right)\times\\
&\left.\left[\frac{\dot{B}}{B}\left(p_r+\rho+2\epsilon+4\eta\sigma\right)+\frac{\dot{C}}{C}\left(\rho+p_\perp+\epsilon-2\eta\sigma\right)\right]\right]+\mathscr{Z}_1(t,r)=0, \qquad (6.29)
\end{aligned}$$

and

$$
\begin{aligned}
&\frac{8\eta f_T\left(1+f_T\right)(\sigma A)'}{ABf_R\left(1-f_T\right)}-\frac{4\eta\left(1+f_T\right)f_R'\sigma}{Bf_R^2}-\frac{f_R'}{Bf_R^2}\left[\left(p_r+\rho+\epsilon\right)f_T+p_r+\epsilon+\frac{f}{2}\right]\\
&-\frac{\dot{f}_R}{Af_R^2}\left(q+\epsilon\right)\left(1+f_T\right)-\frac{4}{AB^3C^2f_R\left(1-f_T\right)}\left[f_T\left(1+f_T\right)\Big(6\eta B^2CC'A\sigma\right.\\
&+\left(q+\epsilon\right)B^3C^2\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)+B^2C'AC\left(p_r-p_\perp+\epsilon\right)\Big)+2A\eta\sigma f_Tf_T'B^2C^2\\
&+\frac{B^2C^2}{2}\left[A'f_T\left(1+f_T\right)\left(p_r+\rho+2\epsilon\right)+f_Tf_T'A\left(p_r+\rho+\epsilon\right)\right.\\
&+\left(A\epsilon'+B\dot{\epsilon}\right)f_T\left(1+f_T\right)+\frac{A}{4}f_T\left(1+f_T\right)\left(T'+4p_r'\right)\\
&\left.\left.+Bf_T\left\{\dot{f}_T\left(q+\epsilon\right)+\dot{q}\left(1+f_T\right)\right\}\right]\right]\\
&+\frac{1}{Bf_R}\left[\rho'f_T+\frac{1}{2}\left(f'-f_RR'\right)\right]+\mathscr{Z}_2(t,r)=0,
\end{aligned}
\tag{6.30}
$$

where the dark source terms are

$$
\begin{aligned}
\mathscr{Z}_1(t,r) \;=\;& -\frac{\dot{f}_RA''}{A^2B^2f_R}+\frac{\dot{f}_R\ddot{B}}{Bf_RA^3}-\frac{2f_R'\dot{C}'}{B^2f_RAC}+\frac{2\dot{f}_R\ddot{C}}{Cf_RA^3}+\frac{\dot{f}_Rf_R''}{AB^2f_R^2}-\frac{f_R'\dot{f}_R{}'}{AB^2f_R^2}\\
&-\frac{\left(2B\dot{C}+\dot{B}C\right)\dot{f}_R{}^2}{A^3BCF^2}-\frac{\dot{f}_R\dot{A}\dot{B}}{BA^4f_R}-\frac{2\dot{A}\dot{C}\dot{f}_R}{A^4f_RC}+\frac{\dot{f}_RA'f_R'}{A^2B^2f_R^2}+\frac{2\dot{C}f_R'A'}{A^2B^2f_RC}\\
&+\frac{\left(B'C-2BC'\right)\left(f_RA'-Af_R'\right)\dot{f}_R}{A^2B^3f_R^2C}+\frac{\left(Cf_R'+2C'f_R\right)f_R'\dot{B}}{AB^3f_R^2C},
\end{aligned}
\tag{6.31}
$$

and

$$
\begin{aligned}
\mathscr{Z}_2(t,r) \;=\;& \frac{2\dot{f}_R\dot{C}'}{A^2Bf_RC}-\frac{f_R'A''}{AB^3f_R}+\frac{f_R'\ddot{B}}{A^2B^2f_R}-\frac{2f_R'C''}{B^3f_RC}+\frac{\dot{f}_R\dot{f}_R{}'}{A^2Bf_R^2}-\frac{\ddot{f}_Rf_R'}{A^2Bf_R^2}\\
&+\frac{f_R'^2\left(A'C+2AC'\right)}{AB^3Cf_R^2}-\frac{f_R'\dot{B}\left(\dot{f}_RA+\dot{A}f_R\right)}{A^3B^2f_R^2}+\frac{2\dot{B}\dot{C}f_R'}{A^2B^2f_RC}-\frac{2C'\dot{B}\dot{f}_R}{A^2B^2f_RC}\\
&+\frac{2B'C'f_R'}{B^4f_RC}-\frac{2f_R'\dot{f}_R\dot{C}}{A^2BCf_R^2}+\frac{f_R'\dot{f}_R\dot{A}}{A^3Bf_R^2}+\frac{f_R'A'B'}{AB^4f_R}-\frac{A'\dot{f}_R{}^2}{A^3Bf_R^2}-\frac{2\dot{C}\dot{f}_RA'}{A^3Bf_RC}.
\end{aligned}
\tag{6.32}
$$

The parameters $\mathscr{Z}_1$ and $\mathscr{Z}_2$ are the extra curvature degrees of freedom representing the dark sources that arise automatically from the $f(R,T)$ gravitational field. These terms signify the corrections in the variation of total energy of the self-gravitating relativistic matter across its boundaries as time evolves. Rearranging the dark source terms, we can represent them in the following form:

$$\mathscr{Z}_1 = \frac{1}{A}\left[-\frac{1}{A^2}\left\{\left(\frac{A^2}{f_R}\left[-\frac{f-Rf_R}{2}+\frac{D_{00}}{A^2}\right]\right)_{,0} - \frac{2\dot{A}A}{f_R}\left(-\frac{f-Rf_R}{2}+\frac{D_{00}}{A^2}\right) - \frac{2A'D_{00}}{Bf_R}\right\}\right.$$
$$+\frac{1}{B^2}\left\{\left(\frac{D_{01}}{f_R}\right)_{,1} - \frac{B\dot{B}}{f_R}\left(-\frac{f-Rf_R}{2}+\frac{D_{00}}{A^2}\right)\right.$$
$$\left.\left.-\left(\frac{B'}{B}+\frac{A'}{A}\right)\frac{D_{01}}{f_R} - \frac{B\dot{B}}{f_R}\left(\frac{f-Rf_R}{2}+\frac{D_{11}}{B^2}\right)\right\}\right],$$

and

$$\mathscr{Z}_2 = \frac{1}{B}\left[-\frac{1}{A^2}\left\{\left(\frac{D_{01}}{f_R}\right)_{,0} - \left(\frac{\dot{B}}{B}+\frac{\dot{A}}{A}\right)\frac{D_{01}}{f_R} - \frac{AA'}{f_R}\left(\frac{f-Rf_R}{2}+\frac{D_{11}}{B^2}\right)\right.\right.$$
$$\left.-\frac{AA'}{f_R}\left(-\frac{f-Rf_R}{2}+\frac{D_{00}}{A^2}\right)\right\}$$
$$\left.+\frac{1}{B^2}\left\{\left(\frac{B^2}{f_R}\left[\frac{f-Rf_R}{2}+\frac{D_{11}}{B^2}\right]\right)_{,1} - \frac{2B\dot{B}D_{01}}{A^2f_R} - \frac{2BB'}{f_R}\left(\frac{f-Rf_R}{2}+\frac{D_{11}}{B^2}\right)\right\}\right].$$

The Misner-Sharp mass function [121] representing the total gravitational energy entrapped inside the surface $\Sigma$ bounding the spherical star of radius 'C', is given by

$$M(v)_\Sigma = \left[\frac{C}{2}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)\right]_\Sigma. \tag{6.33}$$

In order to calculate the variation of this mass function through the boundary surface of the collapsing configuration, we define two well-known operators: the proper time derivative

$$D_T = \frac{1}{A}\frac{\partial}{\partial t}, \tag{6.34}$$

and the proper radial derivative $D_C$ (constructed from the radius of the sphere inside $\Sigma$)

$$D_C = \frac{1}{C'}\frac{\partial}{\partial r}. \tag{6.35}$$

The relativistic 4-velocity of the fluid for the corresponding collapse is given by

$$U = D_T C = \frac{\dot{C}}{A}, \tag{6.36}$$

which must be negative to ensure collapse to occur.

Defining new variable $H(t,r) = \frac{C'}{B}$, we obtain from (6.33) and (6.36)

$$H = \left[1 + U^2 - \frac{2M}{C}\right]^{1/2}. \tag{6.37}$$

With the help of the field equations and equations (6.35)-(6.37), the mass variation in the radial direction is found to be

$$D_C M = \frac{C^2}{2f_R}\left[\rho + \epsilon(1+f_T) + \frac{1}{2}(f - Rf_R) + \frac{\zeta_{00}}{A^2} + \frac{U}{H}\left\{(1+f_T)(q+\epsilon) - \frac{\zeta_{01}}{AB}\right\}\right], \tag{6.38}$$

which on integration yields the result

$$M = \frac{1}{2}\int_0^C \frac{C^2}{f_R}\left[\rho + \epsilon(1+f_T) + \frac{1}{2}(f - Rf_R) + \frac{\zeta_{00}}{A^2} + \frac{U}{H}\left\{(1+f_T)(q+\epsilon) - \frac{\zeta_{01}}{AB}\right\}\right] dC. \tag{6.39}$$

The temporal variation of the mass function is obtained as

$$\begin{aligned} D_T M \quad &= -\frac{C^2}{2f_R}\left[U\left\{(1+f_T)(p_r + \epsilon + 4\eta\sigma) + \rho f_T + \frac{1}{2}(f - Rf_R) + \frac{\zeta_{11}}{B^2}\right\}\right. \\ &\left. + H\left\{(1+f_T)(q+\epsilon) - \frac{\zeta_{01}}{AB}\right\}\right]. \end{aligned} \tag{6.40}$$

In order to determine the variation of the physical parameters of the self-gravitating system in course of time, we now consider a suitable perturbation scheme.

## 6.5 $f(R,T)$ function and Perturbation Scheme

To find the approximate solution of a differential equation, very often one uses the so-called 'perturbation method'. This technique has been successfully applied to analyze the dynamics of gravitational collapse by several authors [23, 28, 29, 64, 24, 25, 26, 19, 98, 100, 208, 46, 47, 48, 52, 106]. We use this technique to analyze the evolution of the stellar object of our study under the effect of $f(R,T)$ model of gravity. To apply the perturbation theory we assume that initially the matter is in a state of hydrostatic equilibrium, so that the initial values of the parameters depend only on the radial coordinate. But in course of time the perturbed quantities gather both radial and time dependence. The radial heat flow $q$ is of the order of $\varepsilon$ ($0 < \varepsilon \ll 1$) [23], (where $\epsilon$ and $\varepsilon$ are different quantities). We consider a combination of the Starobinsky model [92] and a linear term of the trace $T$ as the $f(R,T)$ model [93], which can be written as

$$f(R,T) = R + \alpha R^2 + \lambda T, \tag{6.41}$$

where $\alpha$ has positive real values, while $\lambda$ is a coupling parameter, and $\lambda T$ represents the extent of modification to the Starobinsky $f(R)$ gravity. Thus the metric functions and the physical parameters may be written as follows:

$$\begin{aligned}
A(t,r) &= A_0(r) + \varepsilon D(t) a(r), && (6.42)\\
B(t,r) &= B_0(r) + \varepsilon D(t) b(r), && (6.43)\\
C(t,r) &= C_0(r) + \varepsilon D(t) \bar{c}(r), && (6.44)\\
\rho(t,r) &= \rho_0(r) + \varepsilon \bar{\rho}(t,r), && (6.45)\\
p_r(t,r) &= p_{r0}(r) + \varepsilon \bar{p}_r(t,r), && (6.46)\\
p_\perp(t,r) &= p_{\perp 0}(r) + \varepsilon \bar{p}_\perp(t,r) && (6.47)\\
\epsilon(t,r) &= \varepsilon \bar{\epsilon}(t,r) && (6.48)\\
m(t,r) &= m_0(r) + \varepsilon \bar{m}(t,r), && (6.49)\\
R(t,r) &= R_0(r) + \varepsilon D_1(t) e_1(r), && (6.50)\\
T(t,r) &= T_0(r) + \varepsilon D_2(t) e_2(r), && (6.51)\\
f(R,T) &= [R_0(r) + \alpha R_0^2(r) + \lambda T_0] + \varepsilon D_1(t) e_1(r) &&\\
&\quad \times [1 + 2\alpha R_0(r)] + \varepsilon \lambda D_2(t) e_2(r), && (6.52)\\
f_R &= [1 + 2\alpha R_0(r)] + 2\alpha\varepsilon D_1(t) e_1(r), && (6.53)\\
f_T &= \lambda, && (6.54)\\
\Theta_1(t,r) &= \varepsilon \bar{\Theta}_1, && (6.55)\\
\sigma(t,r) &= \varepsilon \bar{\sigma}(t,r), && (6.56)\\
q(t,r) &= \varepsilon \bar{q}(t,r). && (6.57)
\end{aligned}$$

For $\bar{c}(r) = 0$, we get the shear-free case. Without loss of generality, we consider $C_0(r) = r$ as the Schwarzschild coordinate. In the above equations, $R_0$ represents the static part of the Ricci scalar which has the following form

$$R_0(r) = -\frac{2}{rB_0^2}\left(\frac{1}{r} - \frac{2B_0'}{B_0} + \frac{2A_0'}{A_0}\right) - \frac{2}{B_0^2}\left(\frac{A_0''}{A_0} - \frac{A_0'B_0'}{A_0B_0}\right) + \frac{2}{r^2}, \tag{6.58}$$

while the value of perturbed part of Ricci scalar is given by

$$\begin{aligned} De \;=\; & \frac{2\ddot{D}}{A_0^2}\left(\frac{b}{B_0} + \frac{\bar{c}}{r}\right) + D\left[-\frac{2}{B_0^2}\left\{\frac{A_0'}{A_0}\left(\frac{2\bar{c}'}{r} - \frac{b'}{B_0}\right) - \frac{B_0'}{B_0}\left(\frac{2\bar{c}'}{r} + \frac{a'}{A_0}\right)\right.\right. \\ & \left.\left. + \frac{2}{r}\left(\frac{a'}{A_0} - \frac{b'}{B_0} + \frac{\bar{c}}{r}\right) + \frac{a''}{A_0} + \frac{\bar{c}''}{r}\right\}\right], \end{aligned} \tag{6.59}$$

where we have assumed that the perturbed quantities $D_1 = D_2 = D$, and $e_1 = e_2 = e$. The matter content of the spherical collapsing configuration under static conditions as well as non-static equilibrium phases, are given by

$$m_0 = \frac{r}{2}\left(1 - \frac{1}{B_0^2}\right), \tag{6.60}$$

$$\bar{m}(t,r) = \frac{D(t)}{B_0^2}\left[r\left(\frac{b}{B_0} - \bar{c}'\right) + \frac{\bar{c}}{2}\left(B_0^2 - 1\right)\right]. \tag{6.61}$$

Let $Z = 1 + 2\alpha R_0$, $Z_{R_0} = \frac{\partial Z}{\partial R_0}$, and $Z_{R_0R_0} = \frac{\partial^2 Z}{\partial R_0^2}$. Using these symbols, the static configuration of $f(R,T)$ field equations is obtained in the form

$$\frac{1}{r^2}\left(1 - \frac{1}{B_0^2}\right) + \frac{2B_0'}{rB_0^3} = \frac{1}{Z}\left[\rho_0 + \frac{\alpha R_0^2 - \lambda T_0}{2} + \frac{D_{00}^{(S)}}{A_0^2}\right], \tag{6.62}$$

$$\begin{aligned} \frac{1}{r^2}\left(\frac{1}{B_0^2} - 1\right) + \frac{2A_0'}{A_0B_0^2 r} = & \frac{1}{Z}\left[p_{r0} + \lambda\left(\rho_0 + p_{r0}\right)\right. \\ & \left. - \frac{\alpha R_0^2 - \lambda T_0}{2} + \frac{D_{11}^{(S)}}{B_0^2}\right], \end{aligned} \tag{6.63}$$

$$\begin{aligned} \frac{1}{B_0^2}\left[\frac{A_0''}{A_0} + \frac{A_0'}{A_0 r} - \frac{A_0'B_0'}{A_0B_0} - \frac{B_0'}{B_0 r}\right] = & \frac{1}{Z}\left[p_{\perp 0} + \lambda\left(\rho_0 + p_{\perp 0}\right)\right. \\ & \left. - \frac{\alpha R_0^2 - \lambda T_0}{2} + \frac{D_{22}^{(S)}}{r^2}\right], \end{aligned} \tag{6.64}$$

where

$$D_{00}^{(S)} = \frac{A_0^2}{B_0^2}\left[Z_{R_0}\left\{R_0'' - R_0'\left(\frac{B_0'}{B_0} - \frac{2}{r}\right)\right\} + Z_{R_0R_0}\left(R_0'\right)^2\right], \tag{6.65}$$

$$D_{11}^{(S)} = -Z_{R_0}R_0'\left(\frac{A_0'}{A_0} + \frac{2}{r}\right), \tag{6.66}$$

$$D_{22}^{(S)} = \frac{r^2}{B_0^2}\left[Z_{R_0}\left\{R_0'\left(\frac{B_0'}{B_0} - \frac{A_0'}{A_0} - \frac{1}{r}\right) - R_0''\right\} - Z_{R_0R_0}\left(R_0'\right)^2\right]. \tag{6.67}$$

In the static configuration, the first dynamical equation is identically satisfied, while the second equation is given by

$$\begin{aligned} &-\frac{Z_{R_0}R_0'}{B_0Z^2}\left\{p_{r_0}\left(1+\lambda\right) + \rho_0\lambda + \frac{R_0 + \alpha R_0^2 + \lambda T_0}{2}\right\} + \frac{4\lambda\left(1+\lambda\right)}{B_0Zr\left(-1+\lambda\right)}\left(p_{r_0} - p_{\perp 0}\right) \\ &+\frac{\lambda\left(1+\lambda\right)T_0'}{2\left(-1+\lambda\right)B_0Z} + \frac{2\lambda}{\left(-1+\lambda\right)B_0Z}\frac{1}{B_0Z}\left\{\frac{A_0'}{A_0}\left(p_{r_0} + \rho_0\right)\left(1+\lambda\right) + p_{r_0}'\left(1+\lambda\right) + \rho_0'\lambda\right\} \\ &+\mathscr{Z}_2^{(S)} = 0, \end{aligned} \tag{6.68}$$

where $\mathscr{Z}_2^{(S)}$ represents the static part of $\mathscr{Z}_2(t,r)$ and has the following form:

$$\mathscr{Z}_2^{(S)} = \frac{Z_{R_0}R_0'}{B_0^3Z}\left[-\frac{A_0''}{A_0} + \frac{Z_{R_0}R_0'\left(A_0'r + 2A_0\right)}{A_0rZ} + \frac{2B_0'}{rB_0} + \frac{A_0'B_0'}{A_0B_0}\right]. \tag{6.69}$$

After the application of the perturbation scheme, the perturbed form of the first dynamical equation is

$$\begin{aligned} &-\frac{2\lambda\dot{\bar{\rho}}}{\left(-1+\lambda\right)A_0Z} + \frac{\left(\bar{q}+\bar{\epsilon}\right)\left(1+\lambda\right)}{B_0Z}\left[\frac{Z_{R_0}R_0'}{Z} - 4\frac{\lambda}{\left(-1+\lambda\right)}\left(\frac{A_0'}{A_0} + \frac{1}{r}\right)\right] \\ &-\frac{2\lambda\left(1+\lambda\right)}{\left(-1+\lambda\right)Z}\left[\frac{\bar{\epsilon}' + \bar{q}'}{B_0} + \frac{\dot{\bar{\epsilon}}}{A_0}\right] + \dot{D}\left[\frac{2\alpha e}{Z^2A_0}\left(\rho_0 - \frac{R_0 + \alpha R_0^2 + \lambda T_0}{2}\right)\right. \\ &\left.-\frac{2\lambda\left(1+\lambda\right)}{\left(-1+\lambda\right)ZA_0}\left\{\frac{b}{B_0}\left(p_{r_0} + \rho_0\right) + \frac{2\bar{c}}{r}\left(\rho_0 + p_{\perp 0}\right) - \frac{e}{4}\right\} + \frac{\lambda e}{2A_0Z} + \mathscr{Z}_1^{(P)}\right] = 0, \end{aligned} \tag{6.70}$$

where $\mathscr{Z}_1^{(P)}$ is given in the Appendix.

The perturbed form of the second dynamical equation is

$$\frac{4\eta(1+\lambda)}{B_0 Z}\left[\frac{2\lambda}{(-1+\lambda)}\left(\bar{\sigma}'+\frac{\bar{\sigma}A_0'}{A_0}+\frac{3\bar{\sigma}}{r}\right)-\frac{\bar{\sigma}Z_{R_0}R_0'}{Z}\right]-\frac{Z_{R_0}R_0'}{B_0Z^2}\left[\left(\bar{p_r}+\bar{\epsilon}\right)(1+\lambda)+\lambda\bar{\rho}\right]$$
$$+\frac{\lambda e'}{2B_0Z}-\frac{\alpha e R_0'}{B_0 Z}+\frac{eZ_{R_0}R_0'}{2B_0Z}+\frac{2\lambda}{(-1+\lambda)}\left[\frac{2(1+\lambda)}{B_0Zr}\left(\bar{p_r}-\bar{p_\perp}+\bar{\epsilon}\right)+\frac{A_0'}{A_0B_0Z}\left(\bar{p_r}+\bar{\rho}+2\bar{\epsilon}\right)\right.$$
$$\left.+\frac{(1+\lambda)}{A_0B_0Z}\left\{A_0\left(\bar{p_r}'+\bar{\epsilon}'\right)+B_0\left(\dot{\bar{\epsilon}}+\dot{\bar{q}}\right)\right\}+\frac{\bar{\rho}'\lambda}{B_0Z}\right]+\frac{\ddot{D}bZ_{R_0}R_0'}{A_0^2B_0^2Z}$$
$$+D\left[-\frac{Z_{R_0}R_0'}{B_0Z^2}\frac{e}{2}(Z+\lambda)-\frac{2\alpha e'}{B_0Z^2}\left(\rho_0\lambda+p_{r_0}(1+\lambda)+\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right)\right.$$
$$+\frac{2\lambda}{(-1+\lambda)}\left[\frac{2\bar{c}'}{B_0Zr}(1+\lambda)\left(p_{r_0}-p_{\perp 0}\right)+\frac{(1+\lambda)}{A_0B_0Z}\left[a'\left(p_{r_0}+\rho_0\right)+ap_{r_0}'\right]\right.$$
$$\left.\left.+\frac{a}{A_0B_0Z}\left\{\rho_0'\lambda-\frac{ZR_0'}{2}-\frac{(R_0'+2\alpha R_0R_0'+\lambda T_0')}{2}\right\}\right]+\frac{e'\lambda(1+\lambda)}{2B_0Z(-1+\lambda)}+\mathscr{Z}_2^{(P)}\right]=0, \tag{6.71}$$

where $\mathscr{Z}_2^{(P)}$ is given in the Appendix. From (6.70), integrating with respect to $t$, we obtain,

$$\bar{\rho}=\frac{(-1+\lambda)}{2\lambda}\frac{A_0}{B_0}\left[\frac{Z_{R_0}R_0'}{Z}-\frac{4\lambda}{(-1+\lambda)}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right](1+\lambda)\int\left(\bar{q}+\bar{\epsilon}\right)dt$$
$$-(1+\lambda)\frac{A_0}{B_0}\int\left(\bar{\epsilon}'+\bar{q}'\right)dt-(1+\lambda)\bar{\epsilon}+D\frac{(-1+\lambda)}{2\lambda}\left[\frac{2\alpha e}{Z}\left(\rho_0-\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right)\right.$$
$$\left.-\frac{2\lambda(1+\lambda)}{(-1+\lambda)}\left\{\frac{b}{B_0}\left(p_{r_0}+\rho_0\right)+\frac{2\bar{c}}{r}\left(\rho_0+p_{\perp 0}\right)-\frac{e}{4}\right\}+\frac{\lambda e}{2}+A_0Z\mathscr{Z}_1^{(P)}\right]. \tag{6.72}$$

The perturbed quantities $\bar{\rho}$ and $\bar{p}_r$ are related through the ratio of specific heats, if we consider the second law of thermodynamics and assume a Harrison-Wheeler type equation of state [147], expressed in the following form:

$$\bar{p}_r = \frac{\Gamma p_{r0}}{\rho_0+p_{r_0}}\bar{\rho}, \tag{6.73}$$

where $\Gamma$ is the adiabatic index which determines the change of pressure for a given change in density. We assume that $\Gamma$ remains constant while the collapse progresses.

Now substituting $\bar{\rho}$ from (6.72) in (6.73), we obtain

$$\bar{p}_r = \frac{\Gamma p_{r_0}}{\rho_0 + p_{r_0}} \left[ \frac{(-1+\lambda)}{2\lambda} \frac{A_0}{B_0} \left[ \frac{Z_{R_0} R_0'}{Z} - \frac{4\lambda}{(-1+\lambda)} \left( \frac{A_0'}{A_0} + \frac{1}{r} \right) \right] (1+\lambda) \int (\bar{q} + \bar{\epsilon})\, dt \right.$$
$$- \frac{A_0}{B_0} (1+\lambda) \int (\bar{\epsilon}' + \bar{q}')\, dt - (1+\lambda)\, \bar{\epsilon} + D \frac{(-1+\lambda)}{2\lambda} \left[ \frac{2\alpha e}{Z} \left( \rho_0 - \frac{R_0 + \alpha R_0^2 + \lambda T_0}{2} \right) \right.$$
$$\left. \left. - \frac{2\lambda (1+\lambda)}{(-1+\lambda)} \left\{ \frac{b}{B_0} (p_{r_0} + \rho_0) + \frac{2\bar{c}}{r} (\rho_0 + p_{\perp 0}) - \frac{e}{4} \right\} + \frac{\lambda e}{2} + A_0 Z \mathscr{Z}_1^{(P)} \right] \right]. \quad (6.74)$$

Considering the perturbed variant of the field equation (6.24), we obtain the following form of the perturbed term $\bar{p}_\perp$:

$$\bar{p}_\perp = - \frac{\ddot{D} Z \left( \bar{c} + \frac{br}{B_0} \right)}{A_0^2 r (1+\lambda)} - \frac{\bar{\rho}\lambda}{1+\lambda} + 2\eta\bar{\sigma} + D H_\perp, \quad (6.75)$$

where $H_\perp$ is given in the Appendix. Substituting (6.72), (6.74), and (6.75), in (6.71), we get an equation of the form

$$\ddot{D} - Q(r) D = G(r,t), \quad (6.76)$$

where $Q(r)$ and $G(r,t)$ are given in the Appendix. The terms in (6.76) are chosen to ensure that $Q(r)$ remains positive throughout the evolution. The solution of this differential equation is given by

$$D = \frac{1}{2\sqrt{Q}} \left[ e^{\sqrt{Q}t} \int G e^{-\sqrt{Q}t} dt - e^{-\sqrt{Q}t} \int G e^{\sqrt{Q}t} dt \right]. \quad (6.77)$$

The explicit form of equation (6.76) is given in equation (6.132) in the appendix.

### 6.5.1 Conditions of instability

We now examine the instability conditions of the collapsing system by analysing the Newtonian and post Newtonian regimes of the collapse equations. This analysis also reveals the significance of the adiabatic index $\Gamma$ in the collapse dynamics.

#### Newtonian Approximation

To deal with the Newtonian Approximation of the collapsing configuration, let us consider $A_0 = 1$, $B_0 = 1$. Along with this we assume that $\rho_0 \gg p_{r_0}$, and $\rho_0 \gg p_{\perp 0}$.

Putting these assumptions in the explicit form (6.132) of equation (6.76), we find that the range of stability of the collapsing matter is specified by the following inequality:

$$\Gamma < \frac{ZH_1\left(-1+\lambda\right)}{2\lambda p_{r_0}\left(\frac{\bar{\rho}_N}{\rho_0}\right)'\left(1+\lambda\right)}, \tag{6.78}$$

where $H_1$ is given in the appendix. The collapse will become unstable unless this condition is satisfied. We then have

$$\begin{aligned}\bar{\rho}_N =& \frac{(-1+\lambda)}{2\lambda}\left[\frac{Z_{R_0}R_0'}{Z} - \frac{4\lambda}{(-1+\lambda)}\frac{1}{r}\right](1+\lambda)\int\left(\bar{q}+\bar{\epsilon}\right)dt \\ &+D\frac{(-1+\lambda)}{2\lambda}\left[\frac{2\alpha e}{Z}\left(\rho_0 - \frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right) - (1+\lambda)\int\left(\bar{\epsilon}'+\bar{q}'\right)dt - (1+\lambda)\,\bar{\epsilon}\right. \\ &\left.-\frac{2\lambda\left(1+\lambda\right)}{(-1+\lambda)}\left[\rho_0\left(b+\frac{2\bar{c}}{r}\right) - \frac{e}{4}\right] + \frac{\lambda e}{2} + Z\mathscr{L}_{1_N}^{(P)}\right].\end{aligned} \tag{6.79}$$

To ensure the positivity on the right hand side of the inequality (6.78), we need the following constraints:

$$\frac{2\lambda}{(-1+\lambda)}\left(\bar{\sigma}'+\frac{2\bar{\sigma}}{r}\right)<\frac{\bar{\sigma}Z_{R_0}R_0'}{Z}, \tag{6.80}$$

$$\bar{\epsilon}(1+\lambda)+\lambda\bar{\rho}_N>0, \tag{6.81}$$

$$\frac{Z_{R_0}R_0'}{Z^2}>\frac{2\lambda}{(-1+\lambda)}\frac{2}{Zr}, \tag{6.82}$$

$$e^{\sqrt{Q}t}\int Ge^{-\sqrt{Q}t}dt>e^{-\sqrt{Q}t}\int Ge^{\sqrt{Q}t}dt, \tag{6.83}$$

$$G+\frac{\sqrt{Q}}{2}\left[e^{\sqrt{Q}t}\int Ge^{-\sqrt{Q}t}dt-e^{-\sqrt{Q}t}\int Ge^{\sqrt{Q}t}dt\right]<0, \tag{6.84}$$

$$\frac{2(\bar{c}+br)}{r^2}+\frac{(-1+\lambda)}{2\lambda}\frac{bZ_{R_0}R_0'}{Z}>0, \tag{6.85}$$

$$0<p_{r_0}<p_{\perp 0}, \tag{6.86}$$

$$\bar{\epsilon}'+\dot{\bar{\epsilon}}+\dot{\bar{q}}<0, \tag{6.87}$$

$$\frac{\lambda\bar{\rho}_N'}{Z}<0, \tag{6.88}$$

$$\lambda\rho_0+\frac{R_0+\alpha R_0^2+\lambda T_0}{2}>0, \tag{6.89}$$

$$\frac{Z_{R_0}R_0'e(Z+\lambda)}{2Z^2}>0, \tag{6.90}$$

$$\frac{a'(1+\lambda)\rho_0}{Z}+\frac{\lambda e'}{2Z}-\frac{\alpha eR_0'}{Z}+\frac{eZ_{R_0}R_0'}{2Z}<0, \tag{6.91}$$

$$\rho_0'\lambda-\frac{ZR_0'}{2}-\frac{(R_0'+2\alpha R_0R_0'+\lambda T_0')}{2}<0, \tag{6.92}$$

$$\mathscr{Z}_{2_N}^{(P)}<0, \tag{6.93}$$

$$\left(\frac{\bar{\rho}_N}{\rho_0}\right)'>0. \tag{6.94}$$

### Post-Newtonian Approximation

For the Post-Newtonian Approximation, we consider $A_0=1-\frac{m_0}{r}$, and $B_0=1+\frac{m_0}{r}$. So, we have

$$\frac{A_0}{B_0}=\frac{r-m_0}{r+m_0},\qquad \frac{A_0'}{A_0}=\frac{m_0}{r(r-m_0)},\qquad \text{and}\qquad \frac{B_0'}{B_0}=-\frac{m_0}{r(r+m_0)}.$$

Applying these conditions to equation (6.132), we arrive at the stability criterion

$$\Gamma < \frac{(-1+\lambda)\,Z\,(r+m_0)\,(\rho_0+p_{r_0})\,H_2}{2\lambda p_{r_0}\left\{(1+\lambda)\,r\left(\bar{\rho}'_{pN}-\frac{\bar{\rho}_{pN}\rho'_0}{(\rho_0+p_{r_0})}\right)-\bar{\rho}_{pN}\left(\frac{(-1+\lambda)}{2\lambda}\frac{Z_{R_0}R'_0 r(1+\lambda)}{Z}-2\,(1+\lambda)-\frac{m_0}{(r-m_0)}\right)\right\}}, \tag{6.95}$$

where $H_2$ is given in the appendix. We also have

$$\begin{aligned}\bar{\rho}_{pN} =& \frac{(-1+\lambda)}{2\lambda}\frac{(r-m_0)}{(r+m_0)}\left[\frac{Z_{R_0}R'_0}{Z}-\frac{4\lambda}{(-1+\lambda)}\frac{1}{(r-m_0)}\right](1+\lambda)\int(\bar{q}+\bar{\epsilon})\,dt\\ &-\frac{(r-m_0)}{(r+m_0)}(1+\lambda)\int\left(\bar{\epsilon}'+\bar{q}'\right)dt-(1+\lambda)\bar{\epsilon}+\frac{D\,(\lambda-1)}{2\lambda}\times\left[\frac{2\alpha e}{Z}\left(\rho_0-\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right)\right.\\ &\left.-\frac{2\lambda\,(1+\lambda)}{(-1+\lambda)}\left\{\frac{br}{r+m_0}(p_{r_0}+\rho_0)+\frac{2\bar{c}}{r}(\rho_0+p_{\perp 0})-\frac{e}{4}\right\}+\frac{\lambda e}{2}+\left(1-\frac{m_0}{r}\right)Z\mathscr{Z}^{(P)}_{1_{pN}}\right].\end{aligned} \tag{6.96}$$

To ensure that the right hand side of the inequality (6.95) remains positive, we impose at the following constraints:

$$\left(\bar{\rho}'_{pN} - \frac{\bar{\rho}_{pN}\rho'_0}{(\rho_0 + p_{r_0})}\right) > \bar{\rho}_{pN}\left(\frac{(-1+\lambda)}{2\lambda}\frac{Z_{R_0}R'_0}{Z} - \frac{2}{r} - \frac{m_0}{r(r-m_0)(1+\lambda)}\right), \quad (6.97)$$

$$r + m_0 > 0, \quad (6.98)$$

$$\rho_0 + p_{r_0} > 0, \quad (6.99)$$

$$\frac{2\lambda}{(-1+\lambda)}\left(\bar{\sigma}' + \frac{m_0\bar{\sigma}}{r(r-m_0)} + \frac{2\bar{\sigma}}{r}\right) < \frac{\bar{\sigma}Z_{R_0}R'_0}{Z}, \quad (6.100)$$

$$\bar{\epsilon}(1+\lambda) + \lambda\bar{\rho}_{pN} > 0, \quad (6.101)$$

$$\frac{Z_{R_0}R'_0 r}{Z^2} > \frac{2\lambda}{(-1+\lambda)}\frac{2}{Z}, \quad (6.102)$$

$$e^{\sqrt{Q}t}\int Ge^{-\sqrt{Q}t}dt > e^{-\sqrt{Q}t}\int Ge^{\sqrt{Q}t}dt, \quad (6.103)$$

$$G + \frac{\sqrt{Q}}{2}\left[e^{\sqrt{Q}t}\int Ge^{-\sqrt{Q}t}dt - e^{-\sqrt{Q}t}\int Ge^{\sqrt{Q}t}dt\right] < 0, \quad (6.104)$$

$$\frac{2r\left(\bar{c} + \frac{br^2}{r+m_0}\right)}{(r+m_0)(r-m_0)^2} + \frac{(-1+\lambda)}{2\lambda}\frac{bZ_{R_0}R'_0 r^4}{(r^2-m_0^2)^2 Z} > 0, \quad (6.105)$$

$$\frac{\bar{\epsilon}'}{(r+m_0)} + \frac{\dot{\bar{\epsilon}} + \dot{\bar{q}}}{(r-m_0)} < 0, \quad (6.106)$$

$$\frac{m_0}{(r^2-m_0^2)Z}(2\bar{\epsilon} + \bar{\rho}_{pN}) < 0, \quad (6.107)$$

$$\frac{\lambda r\bar{\rho}'_{pN}}{(r+m_0)Z} < 0, \quad (6.108)$$

$$\frac{Z_{R_0}R'_0 re(Z+\lambda)}{2Z^2(r+m_0)} > 0, \quad (6.109)$$

$$\lambda\rho_0 + p_{r_0}(1+\lambda) + \frac{R_0 + \alpha R_0^2 + \lambda T_0}{2} > 0, \quad (6.110)$$

$$0 < p_{r_0} < p_{\perp_0}, \quad (6.111)$$

$$\frac{a'r^2(1+\lambda)(p_{r_0}+\rho_0)}{(r^2-m_0^2)Z} + \frac{\lambda e'r}{2Z(r+m_0)} - \frac{\alpha erR'_0}{Z(r+m_0)} + \frac{erZ_{R_0}R'_0}{2(r+m_0)Z} < 0, \quad (6.112)$$

$$\rho'_0\lambda - \frac{ZR'_0}{2} - \frac{(R'_0 + 2\alpha R_0R'_0 + \lambda T'_0)}{2} < 0, \quad (6.113)$$

$$\frac{e'\lambda(1+\lambda)r}{2(-1+\lambda)(r+m_0)Z} < 0, \quad (6.114)$$

$$\mathscr{Z}^{(P)}_{2_{pN}} < 0. \quad (6.115)$$

## 6.6 Transport Equations

To derive the transport equation for an arbitrary fluid, we have to keep in mind that both the energy-momentum tensor $T^{ab}$ and the total particle flux $N^a$ are conserved, and the second law of thermodynamics is obeyed by the entropy flux $S^a$. Mathematically, we have

$$T^{ab}_{;b} = 0, \qquad N^a_{;a} = 0, \qquad S^a_{;a} \geq 0.$$

There exists a unique timelike eigenvector $u^a_E$ to $T^{ab}$, and another timelike vector $u^a_N$ parallel to $N^a$. In equilibrium, a rest-frame can be defined in which these eigenvectors, and $S^a$ are parallel to each other. However they are not so when the system deviates from the equilibrium state [127].

Utilising the above three requirements for the energy-momentum tensor, the particle flux and the entropy flux, the transport equation is obtained as

$$\tau h^{ab} u^c q_{b;c} + q^a = -K h^{ab}(T_{1,b} + T_1 a_b) - \frac{1}{2} K T_1^2 \left( \frac{\tau u^b}{K T_1^2} \right)_{;b} q^a. \tag{6.116}$$

where, $\tau$ is the relaxation time (within which the system reverts back to the equilibrium state after being disturbed), $K$ is the thermal conductivity, $T_1$ is the temperature, $h^{ab}$ is the projection tensor, $a_b$ is the acceleration vector, $u^a$ is the four-velocity and $q^a$ is the heat-flux vector. This is the causal heat transport equation derived from the Müller-Israel-Stewart theory [126, 127, 128, 129] for dissipative fluids, and used in [29, 31]. Using the specified definition of the four-velocity in the comoving frame, and radial heat flow, the transport equation can be written as

$$\dot{q} = -\frac{Aq}{\tau} - \frac{K(T_1 A)'}{\tau B} - \frac{qA\Theta_1}{2} - \frac{qKT_1^2}{2\tau} \left( \dot{\overline{\frac{\tau}{KT_1^2}}} \right). \tag{6.117}$$

Rearranging this equation we get

$$\tau \dot{q} = -\frac{qKT_1^2}{2} \left( \dot{\overline{\frac{\tau}{KT_1^2}}} \right) - \tau q \left( \frac{\dot{B}}{2B} + \frac{\dot{C}}{C} \right) - \frac{K(T_1 A)'}{B} - qA. \tag{6.118}$$

which is equation (47) of [31]. This equation can be further simplified as

$$\dot{q} = -\frac{K(T_1 A)'}{\tau B} - qA \left[ \frac{1}{\tau} + \frac{\Theta_1}{2} + \frac{1}{2} D_T \left[ \ln \left( \frac{\tau}{KT_1^2} \right) \right] \right], \tag{6.119}$$

where $D_T = \frac{1}{A}\frac{\partial}{\partial t}$. Finally we can write the transport equation in the form

$$D_T q = -\frac{K(T_1 A)'}{A\tau B} - q\left[\frac{1}{\tau} + \frac{\Theta_1}{2} + \frac{1}{2}D_T\left[\ln\left(\frac{\tau}{KT_1^2}\right)\right]\right]. \tag{6.120}$$

The collapse velocity is given by $U = D_T C = \frac{\dot{C}}{A}$, which must be negative in order to ensure collapse. Considering $D_T U$ as the collapse acceleration, and the expression for the Misner-Sharp mass-energy $M$ contained in a sphere of radius $C$, we obtain from our field equation (7.63),

$$D_T U = -\frac{M}{C^2} - \frac{C}{2f_R}\left[(1+f_T)(p_r+\epsilon+4\eta\sigma) + \rho f_T + \frac{1}{2}(f-RF) + \frac{D_{11}}{B^2}\right] + \frac{A'H}{AB}, \tag{6.121}$$

where $H$ is given in (6.37).

To understand the effects of dissipation on the collapse dynamics, we now combine the transport equation with the dynamical equation, remembering that, $D_C = \frac{1}{C'}\frac{\partial}{\partial r}$. Substituting for $\frac{A'}{A}$ and coupling $D_T q$ with the 2nd dynamical equation (6.30), and utilising all the above relations, we get

$$\begin{aligned}
&(p_r+\rho+2\epsilon-4\eta\sigma)(1-\Lambda)D_T U = (1-\Lambda)F_{grav} + F_{hyd}\\
&- H(q+\epsilon)\left[\frac{(1-f_T)}{2f_R f_T}D_T f_R + \frac{D_T f_T}{1+f_T} + 2\frac{D_T B}{B}\right]\\
&- \frac{H^2(D_C f_R)(1-f_T)}{f_R f_T(1+f_T)}\left[(1+f_T)(p_r+\epsilon+4\eta\sigma) + \rho f_T + \frac{f}{2}\right]\\
&- 2H(q+\epsilon)\left[\frac{U}{C}\right] + \frac{H^2 K D_C T_1}{\tau} + Hq\left[\frac{1}{\tau} + \frac{\Theta_1}{2}\right]\\
&+ H\left[\frac{q}{2}D_T\left[\ln\left(\frac{\tau}{KT_1^2}\right)\right] - D_T\epsilon\right]\\
&+ \frac{H^2 f_R(1-f_T)}{2f_T(1+f_T)}\left[\frac{f_T D_C\rho}{f_R} + \frac{D_C f}{2f_R} - D_C R + \frac{\mathscr{Z}_2(t,r)}{H}\right]
\end{aligned} \tag{6.122}$$

where

$$\Lambda = \frac{KT_1}{\tau}(p_r+\rho+2\epsilon-4\eta\sigma)^{-1}, \tag{6.123}$$

$$\begin{aligned}
F_{grav} &= -(p_r+\rho+2\epsilon-4\eta\sigma)\\
&\times\left[\frac{M}{C^2} + \frac{C}{2f_R}\left[(1+f_T)(p_r+\epsilon+4\eta\sigma) + \rho f_T + \frac{1}{2}(f-RF) + \frac{D_{11}}{B^2}\right]\right],
\end{aligned} \tag{6.124}$$

and

$$F_{hyd} = -H^2 \left[ D_C \left( p_r + \epsilon - 4\eta\sigma + \frac{T}{4} \right) + \frac{2}{C} \left( p_r - p_\perp + \epsilon + 6\eta\sigma \right) \right]. \tag{6.125}$$

A close inspection of (6.122) reveals that the effective inertial mass on the left, and the passive gravitational mass on the right (which is associated with $F_{grav}$), are both reduced by a factor of $\Lambda$, which arises as a result of heat conduction. Hence the ratio of the gravitational mass and the inertial mass remains unchanged, which is precisely what the weak equivalence principle is. This result is in agreement with [133]. The hydrodynamic force term $F_{hyd}$ remains unaffected by the thermal dissipative effects, as its coefficient is unity. It represents the gradient of the effective pressure which involves radiation pressure and also the shear viscosity. It also has a term involving the pressure anistropy, radiation and shear. Since $U$ must be negative for collapse to occur, the term $-2H\left(q+\epsilon\right)\left[\frac{U}{C}\right]$ has a positive contribution, and denotes the matter energy which is escaping in the form of heat-flux and radiation in the free-streaming approximation, which reduces the collapse rate. The terms involving the temperature gradient, the expansion scalar, the thermal conductivity and the relaxation time all arise from the $D_T q$ term coming from the transport equation. For $\Lambda = 1$, the left side of (6.122) vanishes, and the coefficient of $F_{grav}$ reduces to zero. Hence the passive gravitational mass density will no longer affect the collapse. Also, for values of $\Lambda > 1$, the inertial mass density on the left hand side becomes negative, and the sign of $F_{grav}$ also becomes negative. Hence at the critical value of $\Lambda = 1$, we expect a bounce in the system: values of $\Lambda$ less than 1 indicate increase in the inertial mass, which implies collapse of the system, while those greater than 1 implies that the system is undergoing expansion, due to a decrease in the inertial mass [142]. The terms $\frac{H^2 f_R(1-f_T)}{2f_T(1+f_T)}\left[\frac{D_C f}{2f_R} - D_C R + \frac{\mathscr{X}_2(t,r)}{H}\right]$ on the right-hand side of equation (6.122) arise purely from the geometry of the space-time metric and represent the effective dark energy contribution of $f(R,T)$ gravity to the collapsing process.

## 6.7 Summary and Outlook

In this paper, we have considered the most general non-rotating spherically symmetric metric for the interior of the collapsing matter cloud, and have studied the dynamical instability of the dissipative collapse in the $f(R,T)$ theory of gravity. An effective measure of the instability is available from an examination of the limit for the ratio of specific heat, both in the Newtonian approximation as well as the post-Newtonian approximation. Further, we have coupled the dynamical equations with the transport equation in order to examine the effects of dissipation on the evolution of the collapsing system.

Our findings from the analysis of dynamical instability are listed below:

- We see that in case of a non-dissipative, non-radiative fluid with no shear, the differential equation describing the collapse becomes a homogeneous equation and the solution is a negative exponential in $t$ similar to the result obtained in [23]. The entire expression for $G(r, t)$ in (6.76) vanishes in that case.

- In the Newtonian regime, the term $H_1$ will be greater for a shear-free fluid, similar to the result in [26]. It also increases if the heat flux and radiation coefficients are zero. This is because the term involving time derivatives of the heat flux and radiation coefficients contribute negatively to the expression of $H_1$. It further increases in the case of isotropic pressure.

- In the case of Post-Newtonian Approximation, we find that the instability range is greater than that in the Newtonian regime, since the denominator on the right hand side of the inequality involving the ratio of specific heats decreases, and the numerator of the inequality increases. This result is also similar to that obtained in [23]. This conclusion matches with that of Chandrasekhar [145] where it was shown that relativistic effects increase the instability limit for the specific heat ratio.

- Further, in the post-Newtonian regime, we also see that the term $H_2$ increases if all the following quantities vanishes, viz. the shear, radiation, heat flux, and isotropic pressure.

- However it must be mentioned that in order to ensure the positivity of all terms in the expressions for $H_1$ and $H_2$, we have assumed that $p_{r_0} < p_{\perp 0}$, which indeed increases the instability. However the reverse condition, i.e., $p_{r0} > p_{\perp 0}$ will lead to a reduction in the instability compared to the case $p_{r0} < p_{\perp 0}$ [146].

- We also see, from the expression for the total energy trapped inside the fluid surface, that the heat flux and the radiation coefficient contribute negatively towards the total energy, since their coefficient involves $\dot{C}$ which must be negative to ensure collapse of the fluid [146, 24]. This is consistent with the result that both $H_1$ and $H_2$ would have increased if the heat flux and radiation coefficients were put to zero.

Hence it can be concluded that the simultaneous presence of heat flux, radiation and shear decreases the instability range for the adiabatic index, in both Newtonian and relativistic regimes. In other words, the adiabatic index becomes more constrained in presence of dissipative, radiative and viscous effects. Further, the pressure anisotropy also reduces the instability bound. Dynamical instability for a spherical matter distribution with pressure anisotropy, but without any shear, radiation or heat flux was studied in $f(R,T)$ theory [46]. These authors also studied dynamical analysis for collapse of bodies with axial symmetry and with pressure anisotropy in [47], and the effect of the shear-free condition on the adiabatic index in [48] for the same type of matter distribution with anisotropic pressure. Although the dynamics of non-adiabatic charged spherical gravitational collapse of an anisotropic fluid with heat flux have been studied recently in the framework of $f(R,T)$ gravity [142], but the authors did not consider the effect of shear and free-streaming radiation, nor analysed the instability conditions. For the first time, we have studied the dynamical instability for the collapse of a more generalised fluid involving shear, radiation and heat flux and in $f(R,T)$ gravity, considering a linear coupling of the trace of the energy-momentum tensor with the Starobinsky model of $f(R)$, and showed that the instability range for the adiabatic index becomes more restricted in presence of heat flux, shear viscosity and radiation.

We have also derived the transport equation for the given matter in the free streaming approximation. By coupling it to the dynamical equation, we have arrived at the following results:

- Both the active inertial mass and the passive gravitational mass get reduced by the same factor, in agreement with the equivalence principle, and with the works by [29, 133, 140].
- The hydrodynamic force term remains unaffected by the thermal dissipation terms. Also, there is a reduction of mass-energy by the outgoing heat-flux and radiation, which slows down the collapse, as is evident from the negative sign before the term $-2H\left(q+\epsilon\right)\left[\frac{U}{C}\right]$, and the fact that the collapse velocity $U$ must be negative.
- Depending upon whether the value of $\Lambda$ in equation (6.122) is greater than 1 or less than 1, the active inertial mass gets reduced or increased accordingly,

and the system undergoes expansion or collapse. The value $\Lambda = 1$ represents the critical point where the system undergoes a bounce between expansion and collapse. At this point, the passive gravitational mass does not affect the collapsing process [142]. The hydrodynamic force term does not get influenced by changing the values of $\Lambda$ .

There is scope for further investigation of dynamical instability and end results of collapse of the same type of matter distribution in presence of an electromagnetic field, which will be presented in a future work. We are also investigating the thermodynamic behavior of such type of distribution.

# Appendix

$$\begin{aligned}
\mathscr{Z}_1^{(P)} &= \frac{1}{A_0B_0^2Z}\left[-\frac{2\alpha e A_0''}{A_0} - \frac{2\bar{c}' Z_{R_0}R_0'}{r} + \frac{2\alpha e\left\{Z_{R_0R_0}\left(R_0'\right)^2 + Z_{R_0}R_0''\right\}}{Z} - \frac{2\alpha e' Z_{R_0}R_0'}{Z}\right. \\
&\left.+\frac{2\alpha e A_0' Z_{R_0}R_0'}{A_0Z}\right] + \frac{1}{A_0B_0^2Z}\left[\frac{2\alpha e\left(B_0'r - 2B_0\right)\left(ZA_0' - A_0Z_{R_0}R_0'\right)}{A_0B_0Zr}\right. \\
&\left.+\frac{2\bar{c}A_0'Z_{R_0}R_0'}{A_0r} + \frac{bZ_{R_0}R_0'\left[rZ_{R_0}R_0' + 2Z\right]}{B_0Zr}\right], \qquad (6.126) \\
\mathscr{Z}_2^{(P)} &= \frac{2\alpha e' A_0'' + a'' Z_{R_0}R_0'}{A_0B_0^3Z} - \frac{2\bar{c}'' Z_{R_0}R_0'}{B_0^3Zr} + \frac{\left(Z_{R_0}R_0'\right)^2}{A_0B_0^3Z^2r}\left(a'r + \bar{c}'A_0' + \bar{c}'A_0 + 2a\right) \\
&+\frac{2\alpha e' A_0'B_0'}{A_0B_0^4Z} + \frac{4\alpha e' Z_{R_0}R_0'}{A_0B_0^3Z^2r}\left(A_0'r + 2A_0\right) + \frac{2\left(b' + \bar{c}'B_0'\right)Z_{R_0}R_0'}{B_0^4Zr} \\
&+\frac{4\alpha e' B_0'}{B_0^4Zr} + \frac{Z_{R_0}R_0'\left(a'B_0' + b'A_0'\right)}{A_0B_0^4Z}. \qquad (6.127)
\end{aligned}$$

$$\begin{aligned}
H_\perp = \frac{1}{1+\lambda}&\left[\frac{Z}{r^2}\left(\frac{\left(a''r^2 + 2A_0''\bar{c}r + a'r + \bar{c}'A_0'r + \bar{c}A_0'\right)}{A_0B_0^2} + \frac{\bar{c}''r}{B_0}\right.\right. \\
&-\frac{\left(a'B_0'r^2 + b'A_0'r^2 + 2\bar{c}rA_0'B_0'\right)}{A_0B_0^3} - \frac{\left(b'r + \bar{c}'B_0'r + \bar{c}B_0'\right)}{B_0^3} \\
&\left.\left.-\frac{2\bar{c}r}{Z}\left\{\left(1+\lambda\right)p_{\perp 0} + \rho_0\lambda + \frac{R_0 + \alpha R_0^2 + \lambda T_0 - R_0Z}{2} + \frac{D_{22}^{(P)}}{r^2}\right\}\right) - \frac{e\lambda}{2} + \alpha R_0 e\right],
\end{aligned}$$
(6.128)

where

$$D_{22}^{(P)} = \frac{2\alpha R_0' B_0' r^2}{B_0^3} - \frac{2\alpha r}{B_0^2}\left(R_0' + R_0'' r + \frac{R_0' A_0' r}{A_0}\right). \tag{6.129}$$

$$
\begin{aligned}
Q(r) = {} & \frac{1}{\left[\frac{2\lambda}{(-1+\lambda)}\frac{2\left(\bar{c}+\frac{br}{B_0}\right)}{B_0A_0^2r^2}+\frac{bZ_{R_0}R_0'}{A_0^2B_0^2Z}\right]}\Bigg[\frac{Z_{R_0}R_0'}{B_0Z^2}\left(\lambda+(1+\lambda)\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}\right)\frac{(-1+\lambda)}{2\lambda}\Bigg[\frac{2\alpha e}{Z} \\
& \times\left(\rho_0-\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right)-\frac{2\lambda(1+\lambda)}{(-1+\lambda)}\left\{\frac{b}{B_0}(p_{r_0}+\rho_0)+\frac{2\bar{c}}{r}(\rho_0+p_{\perp 0})-\frac{e}{4}\right\} \\
& +\frac{\lambda e}{2}+A_0Z\mathscr{Z}_1^{(P)}\Bigg] \\
& -\Bigg[\frac{2(1+\lambda)}{B_0Zr}\left[\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}+\frac{\lambda}{1+\lambda}\right]\frac{(-1+\lambda)}{2\lambda}\Bigg[\frac{2\alpha e}{Z}\left(\rho_0-\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right) \\
& -\frac{2\lambda(1+\lambda)}{(-1+\lambda)}\left\{\frac{b}{B_0}(p_{r_0}+\rho_0)+\frac{2\bar{c}}{r}(\rho_0+p_{\perp 0})-\frac{e}{4}\right\}+\frac{\lambda e}{2}+A_0Z\mathscr{Z}_1^{(P)}\Bigg] \\
& +\frac{2(1+\lambda)H_\perp}{B_0Zr}-\frac{A_0'}{A_0B_0Z}\left(\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}+1\right)\frac{(-1+\lambda)}{2\lambda}\Bigg[\frac{2\alpha e}{Z}\left(\rho_0-\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right) \\
& -\frac{2\lambda(1+\lambda)}{(-1+\lambda)}\left\{\frac{b}{B_0}(p_{r_0}+\rho_0)+\frac{2\bar{c}}{r}(\rho_0+p_{\perp 0})-\frac{e}{4}\right\}+\frac{\lambda e}{2}+A_0Z\mathscr{Z}_1^{(P)}\Bigg] \\
& -\frac{(1+\lambda)}{B_0Z}\Bigg[\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}\frac{(-1+\lambda)}{2\lambda}\Bigg[\frac{2\alpha e}{Z}\left(\rho_0-\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right) \\
& -\frac{2\lambda(1+\lambda)}{(-1+\lambda)}\left\{\frac{b}{B_0}(p_{r_0}+\rho_0)+\frac{2\bar{c}}{r}(\rho_0+p_{\perp 0})-\frac{e}{4}\right\} \\
& +\frac{\lambda e}{2}+A_0Z\mathscr{Z}_1^{(P)}\Bigg]\Bigg]'-\frac{\lambda}{B_0Z}\frac{(-1+\lambda)}{2\lambda}\Bigg[\frac{2\alpha e}{Z}\left(\rho_0-\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right) \\
& -\frac{2\lambda(1+\lambda)}{(-1+\lambda)}\left\{\frac{b}{B_0}(p_{r_0}+\rho_0)+\frac{2\bar{c}}{r}(\rho_0+p_{\perp 0})-\frac{e}{4}\right\}+\frac{\lambda e}{2}+A_0Z\mathscr{Z}_1^{(P)}\Bigg]'\Bigg]\frac{2\lambda}{(-1+\lambda)} \\
& -\Bigg[-\frac{Z_{R_0}R_0'}{B_0Z^2}\frac{e}{2}(Z+\lambda)-\frac{2\alpha e'}{B_0Z^2}\left(\rho_0\lambda+p_{r_0}(1+\lambda)+\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right) \\
& +\frac{2\lambda}{(-1+\lambda)}\Bigg[\frac{2\bar{c}'}{B_0Zr}(1+\lambda)(p_{r_0}-p_{\perp 0}) \\
& +\frac{(1+\lambda)}{A_0B_0Z}\left[a'(p_{r_0}+\rho_0)+ap_{r_0}'\right]+\frac{\lambda e'}{2B_0Z}-\frac{\alpha eR_0'}{B_0Z}+\frac{eZ_{R_0}R_0'}{2B_0Z} \\
& +\frac{a}{A_0B_0Z}\left\{\rho_0'\lambda-\frac{ZR_0'}{2}-\frac{(R_0'+2\alpha R_0R_0'+\lambda T_0')}{2}\right\}\Bigg]+\frac{e'\lambda(1+\lambda)}{2B_0Z(-1+\lambda)}+\mathscr{Z}_2^{(P)}\Bigg]\Bigg].
\end{aligned}
\tag{6.130}
$$

$$
\begin{aligned}
G(r,t) = & \frac{1}{\left[\frac{2\lambda}{(-1+\lambda)}\frac{2\left(\bar{c}+\frac{br}{B_0}\right)}{B_0 A_0^2 r^2}+\frac{bZ_{R_0}R_0'}{A_0^2 B_0^2 Z}\right]}\left[-\frac{4\eta\,(1+\lambda)}{B_0 Z}\left\{\frac{2\lambda}{(-1+\lambda)}\left(\bar{\sigma}'+\frac{\bar{\sigma}A_0'}{A_0}+\frac{2\bar{\sigma}}{r}\right)\right.\right. \\
& \left.-\frac{\bar{\sigma}Z_{R_0}R_0'}{Z}\right\}+\frac{Z_{R_0}R_0'}{B_0 Z^2}\left[\bar{\epsilon}\,(1+\lambda)\right.+\left(\lambda+(1+\lambda)\,\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}\right) \\
& \times\left\{\frac{(-1+\lambda)}{2\lambda}\frac{A_0}{B_0}\left[\frac{Z_{R_0}R_0'}{Z}-\frac{4\lambda}{(-1+\lambda)}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right](1+\lambda)\int(\bar{q}+\bar{\epsilon})\,dt-(1+\lambda)\,\bar{\epsilon}\right. \\
& \left.\left.-\frac{A_0}{B_0}\,(1+\lambda)\int(\bar{\epsilon}'+\bar{q}')\,dt\right\}\right]-\left[\frac{2\,(1+\lambda)}{B_0 Z r}\left[\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}+\frac{\lambda}{1+\lambda}\right]\right. \\
& \times\left[\frac{(-1+\lambda)}{2\lambda}\frac{A_0}{B_0}\left[\frac{Z_{R_0}R_0'}{Z}-\frac{4\lambda}{(-1+\lambda)}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right](1+\lambda)\int(\bar{q}+\bar{\epsilon})\,dt\right. \\
& \left.-\frac{A_0}{B_0}\,(1+\lambda)\int(\bar{\epsilon}'+\bar{q}')\,dt-(1+\lambda)\,\bar{\epsilon}\right]-\frac{(1+\lambda)\,\bar{\epsilon}'}{B_0 Z}-\frac{(1+\lambda)\,(\dot{\bar{\epsilon}}+\dot{\bar{q}})}{A_0 Z} \\
& -\frac{2\,(1+\lambda)\,\bar{\epsilon}}{B_0 Z r}-\frac{2\bar{\epsilon}A_0'}{A_0 B_0 Z}-\frac{A_0'}{A_0 B_0 Z}\left(\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}+1\right) \\
& \times\left[\frac{(-1+\lambda)}{2\lambda}\frac{A_0}{B_0}\left[\frac{Z_{R_0}R_0'}{Z}-\frac{4\lambda}{(-1+\lambda)}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right](1+\lambda)\int(\bar{q}+\bar{\epsilon})\,dt\right. \\
& \left.-\frac{A_0}{B_0}\,(1+\lambda)\int(\bar{\epsilon}'+\bar{q}')\,dt-(1+\lambda)\,\bar{\epsilon}\right] \\
& -\frac{(1+\lambda)}{B_0 Z}\left[\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}\left[\frac{(-1+\lambda)}{2\lambda}\frac{A_0}{B_0}\left[\frac{Z_{R_0}R_0'}{Z}-\frac{4\lambda}{(-1+\lambda)}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right]\right.\right. \\
& \left.\left.\times(1+\lambda)\int(\bar{q}+\bar{\epsilon})\,dt\;-(1+\lambda)\,\bar{\epsilon}-\frac{A_0}{B_0}\,(1+\lambda)\int(\bar{\epsilon}'+\bar{q}')\,dt\right]\right]' \\
& -\frac{\lambda}{B_0 Z}\left[\frac{(-1+\lambda)}{2\lambda}\frac{A_0}{B_0}\left[\frac{Z_{R_0}R_0'}{Z}-\frac{4\lambda}{(-1+\lambda)}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right](1+\lambda)\int(\bar{q}+\bar{\epsilon})\,dt\right. \\
& \left.\left.\left.-(1+\lambda)\,\bar{\epsilon}-\frac{A_0}{B_0}\,(1+\lambda)\int(\bar{\epsilon}'+\bar{q}')\,dt\right]'\right]\frac{2\lambda}{(-1+\lambda)}\right].
\end{aligned}
\tag{6.131}
$$

The explicit form of equation (6.76) is obtained in the form

$$\frac{4\eta(1+\lambda)}{B_0Z}\left[\frac{2\lambda}{(-1+\lambda)}\left(\bar\sigma'+\frac{\bar\sigma A_0'}{A_0}+\frac{2\bar\sigma}{r}\right)-\frac{\bar\sigma Z_{R_0}R_0'}{Z}\right]-\frac{Z_{R_0}R_0'}{B_0Z^2}\left[\bar\epsilon(1+\lambda)\right.$$

$$+\left(\lambda+(1+\lambda)\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}\right)\left[\frac{(-1+\lambda)}{2\lambda}\frac{A_0}{B_0}\left[\frac{Z_{R_0}R_0'}{Z}-\frac{4\lambda}{(-1+\lambda)}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right]\right.$$

$$\times(1+\lambda)\int(\bar q+\bar\epsilon)\,dt-(1+\lambda)\bar\epsilon-\frac{A_0}{B_0}(1+\lambda)\int(\bar\epsilon'+\bar q')\,dt$$

$$+D\frac{(-1+\lambda)}{2\lambda}\left[\frac{2\alpha e}{Z}\left(\rho_0-\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right)\right.$$

$$\left.\left.\left.-\frac{2\lambda(1+\lambda)}{(-1+\lambda)}\left\{\frac{b}{B_0}(p_{r_0}+\rho_0)+\frac{2\bar c}{r}(\rho_0+p_{\perp 0})-\frac{e}{4}\right\}+\frac{\lambda e}{2}+A_0Z\mathscr{L}_1^{(P)}\right]\right]\right]$$

$$+\frac{2\lambda}{(-1+\lambda)}\left[\frac{2(1+\lambda)\bar\epsilon}{B_0Zr}+\frac{2(1+\lambda)}{B_0Zr}\left[\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}+\frac{\lambda}{1+\lambda}\right]\left[\frac{(-1+\lambda)}{2\lambda}\frac{A_0}{B_0}\left[\frac{Z_{R_0}R_0'}{Z}\right.\right.\right.$$

$$\left.-\frac{4\lambda}{(-1+\lambda)}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right](1+\lambda)\int(\bar q+\bar\epsilon)\,dt-\frac{A_0}{B_0}(1+\lambda)\int(\bar\epsilon'+\bar q')\,dt-(1+\lambda)\bar\epsilon$$

$$+D\frac{(-1+\lambda)}{2\lambda}\left[\frac{2\alpha e}{Z}\left(\rho_0-\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right)\right.$$

$$-\frac{2\lambda(1+\lambda)}{(-1+\lambda)}\left\{\frac{b}{B_0}(p_{r_0}+\rho_0)+\frac{2\bar c}{r}(\rho_0+p_{\perp 0})-\frac{e}{4}\right\}$$

$$\left.\left.+\frac{\lambda e}{2}+A_0Z\mathscr{L}_1^{(P)}\right]\right]-\frac{2(1+\lambda)DH_\perp}{B_0Zr}+\frac{2\ddot D\left(\bar c+\frac{br}{B_0}\right)}{B_0A_0^2r^2}+\frac{2\bar\epsilon A_0'}{A_0B_0Z}$$

$$+\frac{A_0'}{A_0B_0Z}\left(\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}+1\right)\left[\frac{(-1+\lambda)}{2\lambda}\frac{A_0}{B_0}\left[\frac{Z_{R_0}R_0'}{Z}-\frac{4\lambda}{(-1+\lambda)}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right]\right.$$

$$\times(1+\lambda)\int(\bar q+\bar\epsilon)\,dt-\frac{A_0}{B_0}(1+\lambda)\int(\bar\epsilon'+\bar q')\,dt-(1+\lambda)\bar\epsilon$$

$$+D\frac{(-1+\lambda)}{2\lambda}\left[\frac{2\alpha e}{Z}\left(\rho_0-\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right)\right.$$

$$\left.\left.-\frac{2\lambda(1+\lambda)}{(-1+\lambda)}\left\{\frac{b}{B_0}(p_{r_0}+\rho_0)+\frac{2\bar c}{r}(\rho_0+p_{\perp 0})-\frac{e}{4}\right\}+\frac{\lambda e}{2}+A_0Z\mathscr{L}_1^{(P)}\right]\right]$$

$$+\frac{(1+\lambda)\bar\epsilon'}{B_0Z}+\frac{(1+\lambda)(\dot{\bar\epsilon}+\dot{\bar q})}{A_0Z}$$

$$+\frac{(1+\lambda)}{B_0Z}\left[\Gamma\frac{p_{r_0}}{\rho_0+p_{r_0}}\left[\frac{(-1+\lambda)}{2\lambda}\frac{A_0}{B_0}\left[\frac{Z_{R_0}R_0'}{Z}-\frac{4\lambda}{(-1+\lambda)}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right]\right.\right.$$

$$(1+\lambda)\int(\bar q+\bar\epsilon)\,dt-(1+\lambda)\bar\epsilon-\frac{A_0}{B_0}(1+\lambda)\int(\bar\epsilon'+\bar q')\,dt$$ ...continued...

$$
\begin{aligned}
&+ D\frac{(-1+\lambda)}{2\lambda}\left[\frac{2\alpha e}{Z}\left(\rho_0 - \frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right)\right.\\
&\left.\left.\left.- \frac{2\lambda\,(1+\lambda)}{(-1+\lambda)}\left\{\frac{b}{B_0}\left(p_{r_0}+\rho_0\right)+\frac{2\bar{c}}{r}\left(\rho_0+p_{\perp 0}\right)-\frac{e}{4}\right\}+\frac{\lambda e}{2}+A_0 Z\mathscr{Z}_1^{(P)}\right]\right]\right]'\\
&+ \frac{\lambda}{B_0 Z}\left[\frac{(-1+\lambda)}{2\lambda}\frac{A_0}{B_0}\left[\frac{Z_{R_0}R_0'}{Z}-\frac{4\lambda}{(-1+\lambda)}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\right](1+\lambda)\int\left(\bar{q}+\bar{\epsilon}\right)dt\right.\\
&- (1+\lambda)\,\bar{\epsilon}-\frac{A_0}{B_0}(1+\lambda)\int\left(\bar{\epsilon}'+\bar{q}'\right)dt\\
&+ D\frac{(-1+\lambda)}{2\lambda}\left[\frac{2\alpha e}{Z}\left(\rho_0 - \frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right)\right.\\
&\left.\left.\left.- \frac{2\lambda\,(1+\lambda)}{(-1+\lambda)}\left\{\frac{b}{B_0}\left(p_{r_0}+\rho_0\right)+\frac{2\bar{c}}{r}\left(\rho_0+p_{\perp 0}\right)-\frac{e}{4}\right\}+\frac{\lambda e}{2}+A_0 Z\mathscr{Z}_1^{(P)}\right]\right]'\right]\\
&+ \frac{\ddot{D} b Z_{R_0}R_0'}{A_0^2 B_0^2 Z}+D\left[-\frac{Z_{R_0}R_0'}{B_0 Z^2}\frac{e}{2}(Z+\lambda)-\frac{2\alpha e'}{B_0 Z^2}\left(\rho_0\lambda+p_{r_0}(1+\lambda)+\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right)\right.\\
&+ \frac{2\lambda}{(-1+\lambda)}\left[\frac{2\bar{c}'}{B_0 Z r}(1+\lambda)\left(p_{r_0}-p_{\perp 0}\right)+\frac{(1+\lambda)}{A_0 B_0 Z}\left[a'\left(p_{r_0}+\rho_0\right)+a p_{r_0}'\right]+\frac{\lambda e'}{2B_0 Z}\right.\\
&\left.- \frac{\alpha e R_0'}{B_0 Z}+\frac{e Z_{R_0}R_0'}{2B_0 Z}+\frac{a}{A_0 B_0 Z}\left\{\rho_0'\lambda-\frac{Z R_0'}{2}-\frac{\left(R_0'+2\alpha R_0 R_0'+\lambda T_0'\right)}{2}\right\}\right]\\
&\left.+\frac{e'\lambda\,(1+\lambda)}{2B_0 Z\,(-1+\lambda)}+\mathscr{Z}_2^{(P)}\right]=0. \qquad (6.132)
\end{aligned}
$$

$$
\begin{aligned}
H_1 = & -\frac{4\eta\left(1+\lambda\right)}{Z}\left[\frac{2\lambda}{\left(-1+\lambda\right)}\left(\bar{\sigma}'+\frac{2\bar{\sigma}}{r}\right)-\frac{\bar{\sigma}Z_{R_0}R_0'}{Z}\right]+\frac{Z_{R_0}R_0'}{Z^2}\left[\bar{\epsilon}\left(1+\lambda\right)+\lambda\bar{\rho}_N\right] \\
& +\frac{2\lambda}{\left(-1+\lambda\right)}\left[-\frac{2\bar{\epsilon}\left(1+\lambda\right)}{Zr}-\frac{2\lambda\bar{\rho}_N}{Zr}\right. \\
& +\frac{2\left(1+\lambda\right)H_{\perp_N}}{Zr}\frac{1}{2\sqrt{Q}}\left[e^{\sqrt{Q}t}\int Ge^{-\sqrt{Q}t}dt-e^{-\sqrt{Q}t}\int Ge^{\sqrt{Q}t}dt\right] \\
& -\left[G+\frac{\sqrt{Q}}{2}\left[e^{\sqrt{Q}t}\int Ge^{-\sqrt{Q}t}dt-e^{-\sqrt{Q}t}\int Ge^{\sqrt{Q}t}dt\right]\right] \\
& \times\left[\frac{2\left(\bar{c}+br\right)}{r^2}+\frac{\left(-1+\lambda\right)}{2\lambda}\frac{bZ_{R_0}R_0'}{Z}\right]-\frac{\left(1+\lambda\right)}{Z}\left[\bar{\epsilon}'+\dot{\bar{\epsilon}}+\dot{\bar{q}}\right]-\frac{\lambda\bar{\rho}_N'}{Z} \\
& -\frac{1}{2\sqrt{Q}}\left[e^{\sqrt{Q}t}\int Ge^{-\sqrt{Q}t}dt-e^{-\sqrt{Q}t}\int Ge^{\sqrt{Q}t}dt\right]\left[-\frac{Z_{R_0}R_0'e\left(Z+\lambda\right)}{2Z^2}\right. \\
& -\frac{2\alpha e'}{Z^2}\left(\lambda\rho_0+\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right) \\
& +\frac{2\lambda}{\left(-1+\lambda\right)}\left[\frac{2\bar{c}'}{Zr}\left(1+\lambda\right)\left(p_{r_0}-p_{\perp_0}\right)+\frac{a'\left(1+\lambda\right)\rho_0}{Z}+\frac{\lambda e'}{2Z}-\frac{\alpha eR_0'}{Z}\right. \\
& \left.\left.\left.+\frac{eZ_{R_0}R_0'}{2Z}+\frac{a}{Z}\left\{\rho_0'\lambda-\frac{ZR_0'}{2}-\frac{\left(R_0'+2\alpha R_0R_0'+\lambda T_0'\right)}{2}\right\}+\mathscr{Z}_{2_N}^{(P)}\right]\right]\right]. \qquad (6.133)
\end{aligned}
$$

$$
\begin{aligned}
H_2 = & -\frac{4\eta r\left(1+\lambda\right)}{\left(r+m_0\right)Z}\left[\frac{2\lambda}{\left(-1+\lambda\right)}\left(\bar{\sigma}'+\frac{m_0\bar{\sigma}}{r\left(r-m_0\right)}+\frac{2\bar{\sigma}}{r}\right)-\frac{\bar{\sigma}Z_{R_0}R_0'}{Z}\right] \\
& +\frac{Z_{R_0}R_0' r}{\left(r+m_0\right)Z^2}\left[\bar{\epsilon}\left(1+\lambda\right)+\lambda\bar{\rho}_{pN}\right]+\frac{2\lambda}{\left(-1+\lambda\right)}\left[-\frac{2\bar{\epsilon}\left(1+\lambda\right)}{\left(r+m_0\right)Z}-\frac{2\lambda\bar{\rho}_{pN}}{\left(r+m_0\right)Z}\right. \\
& +\frac{2\left(1+\lambda\right)H_{\perp_{pN}}}{\left(r+m_0\right)Z}\frac{1}{2\sqrt{Q}}\left[e^{\sqrt{Q}t}\int Ge^{-\sqrt{Q}t}dt-e^{-\sqrt{Q}t}\int Ge^{\sqrt{Q}t}dt\right] \\
& -\left[G+\frac{\sqrt{Q}}{2}\left[e^{\sqrt{Q}t}\int Ge^{-\sqrt{Q}t}dt-e^{-\sqrt{Q}t}\int Ge^{\sqrt{Q}t}dt\right]\right] \\
& \times\left(\frac{2r\left(\bar{c}+\frac{br^2}{r+m_0}\right)}{\left(r+m_0\right)\left(r-m_0\right)^2}+\frac{\left(-1+\lambda\right)}{2\lambda}\frac{bZ_{R_0}R_0' r^4}{\left(r^2-m_0^2\right)^2 Z}\right) \\
& -\frac{\left(1+\lambda\right)r}{Z}\left[\frac{\bar{\epsilon}'}{\left(r+m_0\right)}+\frac{\dot{\bar{\epsilon}}+\dot{\bar{q}}}{\left(r-m_0\right)}\right]-\frac{m_0}{\left(r^2-m_0^2\right)Z}\left(2\bar{\epsilon}+\bar{\rho}_{pN}\right)-\frac{\lambda r\bar{\rho}_{pN}'}{\left(r+m_0\right)Z} \\
& -\frac{1}{2\sqrt{Q}}\left[e^{\sqrt{Q}t}\int Ge^{-\sqrt{Q}t}dt-e^{-\sqrt{Q}t}\int Ge^{\sqrt{Q}t}dt\right]\left[-\frac{Z_{R_0}R_0' re\left(Z+\lambda\right)}{2Z^2\left(r+m_0\right)}\right. \\
& -\frac{2\alpha e' r}{\left(r+m_0\right)Z^2}\left(\lambda\rho_0+p_{r_0}\left(1+\lambda\right)+\frac{R_0+\alpha R_0^2+\lambda T_0}{2}\right) \\
& +\frac{2\lambda}{\left(-1+\lambda\right)}\left[\frac{2\bar{c}'}{Z\left(r+m_0\right)}\left(1+\lambda\right)\left(p_{r_0}-p_{\perp 0}\right)\right. \\
& +\frac{a' r^2\left(1+\lambda\right)\left(p_{r_0}+\rho_0\right)}{\left(r^2-m_0^2\right)Z}+\frac{\lambda e' r}{2Z\left(r+m_0\right)}-\frac{\alpha erR_0'}{Z\left(r+m_0\right)}+\frac{erZ_{R_0}R_0'}{2\left(r+m_0\right)Z} \\
& +\frac{ar^2}{Z\left(r^2-m_0^2\right)}\left\{\rho_0'\lambda-\frac{ZR_0'}{2}-\frac{\left(R_0'+2\alpha R_0R_0'+\lambda T_0'\right)}{2}\right\}\Bigg]\Bigg] \\
& +\frac{e'\lambda\left(1+\lambda\right)r}{2\left(-1+\lambda\right)\left(r+m_0\right)Z}+\mathscr{Z}_{2_{pN}}^{(P)}\Bigg].
\end{aligned}
\tag{6.134}
$$

”

# Chapter 7

# Singularity and Apparent Horizon

This chapter discusses the junction conditions, time of formation of singularity and apparent horizon, and constraints for black hole formation for collapse of an isotropic spherically symmetric matter ball with radial heat flow in $f(R,T)$ gravity. The publication details of the material in this chapter are given below.

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The paper is quoted below :

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## 7.1 Introduction

The study of gravitational collapse of a massive object has been an area of interest for many years. Depending on the mass of the collapsing object, the final stage of such collapse may lead to the formation of spacetime singularities where normal laws of physics are no longer valid. Over the years various aspects of collapse have been studied by several researchers [18, 19, 58, 63, 20, 23, 24, 25, 26, 28, 29, 64, 21], which include dynamical instability, causal heat transport, as well as the examination of the progress of collapse for different types of matter, which may or may not involve shear, heat flux, free-streaming radiation, and/or anisotropy of pressure. Different combinations of these factors, lead to the difference in the nature of the end state of collapse. There have also been attempts to search for solutions to the Einstein Field Equations for various kinds of matter [27, 28], where the Vaidya metric [148] has been considered to describe the spacetime outside the collapsing matter.

The failure of General Relativity (GR) to account for the observed accelerated expansion of the universe as revealed from the observational data from the Type Ia supernova [3, 4], led scientists to propose an unknown component in the matter-energy sector of the universe, called the dark energy [2]. In addition to this, the observation of the galactic rotation curves also indicate the existence of an entity called "dark matter", which seems to dominate the matter content of the universe. It seems that at extremely large scales, Einstein's theory of gravity is not a suitable option to explain the observed features of the universe. Hence, new theories of modified gravity were developed [69, 70, 8, 71, 72, 73, 74, 75] in order to explain astrophysical phenomena at large scales. One such theory was the $f(R)$ theory in which the Einstein-Hilbert action was generalised by replacing the Ricci scalar $R$ appearing in it, by a function of the Ricci scalar, $f(R)$, which depends only on the geometry of the spacetime and not on the matter content. This modification stemmed from the idea of including the higher order terms of curvature in the action integral, which give rise to the dark energy components. Moreover, the presence of these higher order curvature terms mimics the role of the cosmological constant $\Lambda$ in GR at the current epoch of evolution of the universe (thereby representing the $\Lambda$CDM model), and integrates all the phases of evolution of the universe in a single model [84, 85, 86]. Barraco and Hamity [150] found spherically symmetric solutions in a first order approximation of $f(R)$ theory of gravity, and showed that both the exterior and interior metrics satisfy the junction conditions. They also showed that at least one exterior solution is the Schwarzschild metric. Capozziello, Stabile and Troisi [151] used the existence of Noether symmetry to find spherically symmetric solutions in $f(R)$ theory. They also found classes of exact solutions for spherically symmetric case in $f(R)$ theory, both for constant Ricci scalar $R_0$ as well as $R(r)$ where $r$ is the radial coordinate [152]. More spherically symmetric solutions in $f(R)$ gravity were found using this Noether symmetry approach [153]. Multamaki and Vilja [154] examined static empty space solutions with spherical symmetry in $f(R)$ theory of gravity. Chakrabarti and Banerjee [43] found the time of apparent horizon formation and singularity formation for a perfect fluid collapse in $f(R)$ gravity. Sharif and Kausar [36] found apparent horizons for spherically symmetric perfect fluid collapse in $f(R)$ theory.

The $f(R,T)$ theory, first proposed by Harko et al. [9] is a further modification of GR, which can provide a suitable explanation for this accelerated expansion of the universe. This theory is a further generalisation of the $f(R)$ theory, in which, the Ricci scalar $R$ in the Einstein-Hilbert action is replaced by a function of $R$ and $T$, the latter being the trace of the energy-momentum tensor appearing in the Einstein field equations. The $f(R,T)$ function arises due to quantum effects or due to the existence of exotic imperfect matter fluids. In their paper [9], Harko and collaborators also discussed a few cases for possible choices of the $f(R,T)$ function. A suitable choice is the linear form $R+2\lambda T$ which gives rise to power-law type of scale factors in the corresponding cosmological model. Sahoo et al. [155] showed that $f(R)+\lambda T$ gravity models act as alternatives to cosmic acceleration. The inclusion of the trace $T$ of the energy-momentum tensor in the Einstein-Hilbert action enables one to study the effect of curvature-matter interaction in the evolution of the universe [99, 156, 157, 97, 47, 158, 159]. Moraes and Sahoo constructed wormhole models [160], and also proposed a cosmological scenario from the simplest non-minimal matter-geometry coupling in the $f(R,T)$ theory of gravity [161]. Moraes, Correa and Lobato [162] found solutions for a static wormhole metric in the $f(R,T)$ framework with linearised form of the $f(R,T)$ function. Sharif and Fatima [163] studied the effects of charge on traversable wormhole structure in $f(R,T)$ theory. Zaregonbadi, Farhoudi and Riazi [164] found solutions for a static spherically symmetric spacetime in $f(R,T)$ gravity, and extracted the expressions for the metric components in the case of the galactic halo, using minimal coupling. Amir and Sattar [51] investigated spherically symmetric collapse of a perfect fluid in $f(R,T)$ gravity, and determined the conditions for the formation of the apparent horizon. Abbas and Ahmed [53] studied charged perfect fluid collapse and apparent horizon formation in $f(R,T)$ theory. Yousaf et al. [102, 50] studied the influence of structure scalars obtained by orthogonal splitting of the Riemann tensor on various physical properties of the collapsing matter, such as energy density inhomogeneity, pressure anisotropy and shear viscosity in $f(R,T)$ theory, and provided an insight on how the evolution of the collapsing matter proceeds under the effect of tidal forces and these inhomogeneities. Yousaf [228] considered modelling of a gravastar in cylindrical symmetry in $f(R,T)$ theory, which could be a suitable alternative to a black hole as the final state of the collapse, and studied the effect of electromagnetic field on the mass-energy content of the middle thin shell of the gravastar, and also drew comparisons between its density and pressure, and its proper length and thickness. In a previous work [54], the present authors studied the dynamical stability and heat transport for a collapsing dissipative fluid in $f(R,T)$ gravity.

In this paper, we consider the region outside the collapsing matter to be described by the generalized Vaidya spacetime, which is then used to apply the $f(R,T)$ junction conditions [230] in order to solve the $f(R,T)$ field equations corresponding to the interior spacetime. We assume the geometry to be spherically symmetric so that there is no generation of gravitational waves, and the matter distribution involves pressure isotropy and heat flux. The spherical symmetry can also be used to model realistic gravitational collapse with small deviations only. The exterior region is assumed to be filled with a combination of Type-I and Type-II fluids, similar to what Wang and Wu [165] had considered while working with a generalized Vaidya metric (which was a generalization of the original Vaidya metric [166]). The metric coefficients are assumed to be separable into spatial and temporal parts, similar to the works of Sharif and Abbas [167], and Guha and Banerji [35]. The time of formation of the singularity, and that of the apparent horizon is determined and the nature of the resulting singularity is analysed in consideration of the work of Joshi, Goswami and Dadhich [168]. The time of formation of apparent horizon for different types of collapse was studied previously in GR [169, 170]. Here, a study of the same is carried out in $f(R,T)$ theory.

The organization of this paper is as follows: In Section II we provide a brief introduction to the $f(R,T)$ formalism, and proceed to consider a generalized Vaidya exterior spacetime and determine the field equations for a combination of type-I and type-II fluids in Section III. Section IV consists of the description of the interior spacetime and the field equations for the interior spacetime are determined. In Section V, we enlist the junction conditions for the $f(R,T)$ theory and examine the results that emerge after the application of the junction conditions. In Section VI, the time of formation of singularity is determined by examining the nature of the temporal dependence of the physical radius of the collapsing matter. Section VII deals with the formation of apparent horizon of the collapsing system, the nature of the final singularity, and the restrictions on the parameters governing the evolution of the system so as to ensure black hole formation. This is followed by a summarization and discussion in Section VIII.

## 7.2 The $f(R,T)$ Formalism

The modified Einstein-Hilbert action in $f(R,T)$ gravity, is given by

$$S = \int d^4x\sqrt{-g}\left(\frac{f(R,T)}{16\pi G} + \mathcal{L}_m\right), \tag{7.1}$$

where $g$ is the determinant of the spacetime metric, $G$ is the gravitational constant, $f(R,T)$ is an arbitrary function of the Ricci scalar $R$ and the trace $T$ of the energy-momentum tensor of the matter filling up the spacetime, and $L_m$ is the matter Lagrangian. The field equations of $f(R,T)$ gravity [9] can be written in the form :

$$R_{\mu\nu} = \frac{1}{f_R}\left[(1+f_T)\,T^m_{\mu\nu} - L_m g_{\mu\nu} f_T + \frac{1}{2} g_{\mu\nu} f - D_{\mu\nu}\right], \tag{7.2}$$

where $g_{\mu\nu}$ is the spacetime metric, $R$ is the Ricci scalar, $T$ is the trace of the energy-momentum tensor, $f_R$ and $f_T$ are the derivatives of the $f(R,T)$ function with respect to $R$ and $T$, respectively, $L_m$ is the matter Lagrangian, and $D_{\mu\nu} = (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\, f_R$ which includes the higher order curvature terms, and acts as the source of dark energy.

In order to obtain the form of the field equation (8.100) in $f(R,T)$ gravity, it was assumed by Harko in [9] that the matter Lagrangian was minimally coupled to the metric, that is, it depended only on the components of the metric tensor $g_{\mu\nu}$ but not on the derivatives of $g_{\mu\nu}$. In what follows, we will choose $f(R,T) = R + 2\lambda T$ for our analysis.

## 7.3 The Exterior Spacetime

The boundary of the collapsing matter divides the entire spacetime into an interior region containing the collapsing matter, and an exterior region containing the material particles, photons, and thermal radiation emanating from the star during the collapse. The exterior spacetime in the region outside the boundary of the collapsing body is considered to be described by the generalised Vaidya metric of outgoing radiation, and is given by

$$ds_+^2 = -\left(1 - \frac{2M(v,Y)}{Y}\right) dv^2 - 2dvdY + Y^2 d\Omega^2, \tag{7.3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Here, $v$ is the retarded null coordinate, and $Y$ is the radial coordinate for the exterior region. The parameter $M(v,Y)$ represents the mass-energy content inside a radius $Y$ at a time $v$.

The exterior energy-momentum tensor is assumed to represent a combination of type-I and type-II fluids [165], and is expressed as follows:

$$T^+_{\mu\nu} = \mu l_\mu l_\nu + (\rho + P)\,(l_\mu n_\nu + l_\nu n_\mu) + P g^+_{\mu\nu}, \tag{7.4}$$

where the first term represents a type-I fluid of density $\mu$, which represents null-like matter/radiation photons, and the second and third terms together represent a type-II fluid, with $\rho$ being its density and $P$ being its isotropic pressure, which represents a perfect fluid component which can be timelike/massive particles. For a type-I fluid, the energy-momentum tensor has one timelike eigenvector, while for a type-II fluid, the energy-momentum tensor has a double null eigenvector. A discussion on various types of matters can be found in the book [171]. Here, $g^{+}_{\mu\nu}$ is the metric tensor for the exterior spacetime. The positive sign in the subscript of the line element and the superscript of the energy-momentum tensor denotes the exterior spacetime. Further,

$$l_\mu = \delta^0_\mu, \tag{7.5}$$

$$\text{and} \qquad n_\mu = \frac{1}{2}\left(1 - \frac{2M(v,Y)}{Y}\right)\delta^0_\mu - \delta^1_\mu. \tag{7.6}$$

The trace of the exterior energy-momentum tensor is

$$T^{+} = 6P + 2\rho. \tag{7.7}$$

Invoking the equation of state for pure radiation, $P = \rho/3$, and applying it to equation (8.99), we obtain $T^{+} = 4\rho$.

On account of our choice $f(R,T) = R + 2\lambda T$, we find that $f_R = 1$, $f_T = 2\lambda$ and $D_{\mu\nu} = 0$. In that case, the $f(R,T)$ field equations for the exterior spacetime reduces to :

$$\begin{aligned}
R_{00} &= \frac{1}{Y^2}\left(\left(\frac{\partial^2 M}{\partial Y^2}\right)(2M - Y) - 2\frac{\partial M}{\partial v}\right) \\
&= \left(1 - \frac{2M}{Y}\right)\left((1+2\lambda)\,\rho + 2\lambda L_{m_{ext}} - \frac{f}{2}\right) + (1+2\lambda)\,\mu, \qquad (7.8)\\
R_{11} &= 0, \qquad (7.9)\\
R_{22} &= 2\frac{\partial M}{\partial Y} = Y^2\left[(1+2\lambda)\,P - 2\lambda L_{m_{ext}} + \frac{f}{2}\right], \qquad (7.10)\\
R_{33} &= R_{22}\sin^2\theta, \qquad (7.11)\\
R_{01} &= -\frac{1}{Y}\frac{\partial^2 M}{\partial Y^2} = -\,(1+2\lambda)\,(\rho + P) - \frac{R_{22}}{Y^2}. \qquad (7.12)
\end{aligned}$$

where, $L_{m_{ext}}$ is the matter Lagrangian for the exterior spacetime.

## 7.4 The Interior Spacetime

The interior spacetime is assumed to be of the most general spherically symmetric form, given by

$$ds_-^2 = -A(r,t)^2dt^2 + B(r,t)^2dr^2 + C(r,t)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{7.13}$$

The matter-energy momentum tensor for the interior spacetime is considered to be that of an isotropic fluid undergoing dissipation in the form of heat flux, and is given by

$$T_{\mu\nu}^{(m)-} = \left(\rho_{int} + p\right)u_\mu u_\nu + pg_{\mu\nu}^- + q_\mu u_\nu + q_\nu u_\mu, \tag{7.14}$$

where, $\rho_{int}$ is the interior energy density, $p$ is the isotropic pressure, and the four-velocity is given by $u^\mu = A^{-1}\delta_0^\mu$. We define $\chi^\mu = B^{-1}\delta_1^\mu$, where $\chi^\mu$ is a unit vector in the radial direction. Then the heat flux vector is defined as $q^\mu = q\chi^\mu$ where $q = q(r,t)$ is the heat flux, and the four-velocity satisfies $u^\mu u_\mu = -1$. The trace of the interior energy-momentum tensor is given by

$$T^- = -\rho_{int} + 3p. \tag{7.15}$$

The $f(R,T)$ field equations are given by

$$R_{\mu\nu} = \frac{1}{f_R}\left[\left(1+f_T\right)T_{\mu\nu}^{m-} - L_m g_{\mu\nu}^- f_T + \frac{1}{2}g_{\mu\nu}^- f - D_{\mu\nu}\right], \tag{7.16}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R$ is the Ricci scalar, $T$ is the trace of the energy-momentum tensor, $f_R$ and $f_T$ are the derivatives of the $f(R,T)$ function with respect to $R$ and $T$, respectively, $L_m$ is the interior matter Lagrangian, and

$$D_{\mu\nu} = \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f_R$$

includes the higher order curvature terms, which acts as the source of dark energy.

Just as in the case of the exterior spacetime, the choice $f(R,T) = R + 2\lambda T$, implies that $f_R = 1$, $f_T = 2\lambda$ and $D_{\mu\nu} = 0$. Evidently, the Ricci scalar $R$ and the trace $T$ will be different for the interior and the exterior spacetimes. The interior field equations then take the following form :

$$\begin{aligned}
R_{00} &= -\frac{2\ddot{C}}{C} + \frac{2\dot{A}\dot{C}}{AC} - \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{2AA'C'}{B^2C} + \frac{AA''}{B^2} - \frac{AA'B'}{B^3} \\
&= A^2\left[(1+2\lambda)\,\rho_{int} + 2\lambda L_m - \frac{f}{2}\right], && (7.17) \\
R_{11} &= -\frac{2C''}{C} + \frac{B\ddot{B}}{A^2} + \frac{2B\dot{B}\dot{C}}{A^2C} - \frac{\dot{A}\dot{B}B}{A^3} - \frac{A''}{A} + \frac{2B'C'}{BC} + \frac{A'B'}{AB} \\
&= B^2\left((1+2\lambda)\,p - 2\lambda L_m + \frac{f}{2}\right), && (7.18) \\
R_{22} &= -\frac{CC''}{B^2} + \frac{C\ddot{C}}{A^2} - \frac{C\dot{C}\dot{A}}{A^3} + \frac{C\dot{C}\dot{B}}{A^2B} - \frac{CC'A'}{AB^2} + \frac{CC'B'}{B^3} + \frac{2m(r,t)}{C} \\
&= C^2\left[(1+2\lambda)\,p - 2\lambda L_m + \frac{f}{2}\right], && (7.19) \\
R_{33} &= R_{22}\sin^2\theta, && (7.20) \\
R_{01} &= -\frac{2\dot{C}'}{C} + \frac{2A'\dot{C}}{AC} + \frac{2\dot{B}C'}{BC} \\
&= -\,(1+2\lambda)\,qAB && (7.21)
\end{aligned}$$

where

$$m(r,t) = \frac{C}{2}\left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2}\right). \qquad (7.22)$$

## 7.5 The Junction Conditions

The junction conditions represent the continuity of relevant quantities across the boundary of the collapsing star. In their works, Darmois and Israel had presented and discussed the junction conditions in the context of GR [172, 173]. The continuity of the line element and the extrinsic curvature tensor are the only necessary junction conditions in GR. The consideration of junction conditions depends on the nature of the Einstein-Hilbert action in the various theories of gravity. In the $f(R)$ theory of gravity, in addition to the junction conditions in GR, one has to include the continuation of the Ricci scalar and its derivative. The continuity of the trace part and the trace-free parts of the extrinsic curvature tensor are also to be considered separately. Junction conditions in $f(R)$ gravity were discussed by the authors in [174, 175, 176, 177]. Further, in the $f(R,T)$ theory, the trace of the energy-momentum tensor and the derivative of the trace are two additional quantities required to be matched across the hypersurface $\Sigma$.

The timelike 3D-hypersurface $\Sigma$ which separates the interior and the exterior spacetime, and serves as the boundary of the collapsing matter, is given by

$$ds^2_\Sigma = -d\tau^2 + \mathcal{R}(\tau)^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{7.23}$$

where $(\tau, \theta, \phi)$ represents the coordinates on the hypersurface $\Sigma$, $\mathcal{R}(\tau)$ is the radius of the 2-sphere specified by the angular coordinates.

The junction conditions for the perfect-fluid version of $f(R,T)$ gravity (both in the geometrical representation as well as in a dynamically equivalent scalar-tensor representation) were derived by Rosa [230]. In the particular case of a smooth matching with no thin-shell, those junction conditions are:

$$[g_{\mu\nu}]^+_- = 0, \tag{7.24}$$

$$\left[\tilde{K}_{ij}\right]^+_- = 0, \tag{7.25}$$

$$[K]^+_- = 0, \tag{7.26}$$

$$[R]^+_- = 0, \tag{7.27}$$

$$[T]^+_- = 0, \tag{7.28}$$

$$[\partial_\mu T]^+_- = 0, \tag{7.29}$$

$$[\partial_\mu R]^+_- = 0. \tag{7.30}$$

Here, the Greek indices represent the coordinates of the 4D-spacetimes, while the Latin indices represent the coordinates of the hypersurface. The negative sign signifies the interior spacetime, while the positive sign signifies the exterior spacetime.

The junction condition in (8.125), which is similar to the first Darmois-Israel junction condition [172, 173] can be written as

$$ds^2_- = ds^2_\Sigma = ds^2_+, \tag{7.31}$$

which gives us

$$A^2(r,t)\, dt^2 |_\Sigma = d\tau^2 |_\Sigma = \left(1 - \frac{2M(v,Y)}{Y} + 2\frac{dY}{dv}\right) dv^2 |_\Sigma\,, \tag{7.32}$$

$$\text{and} \qquad C(t, r_\Sigma) = \mathcal{R}_\Sigma(\tau) = Y_\Sigma(\tau)\,. \tag{7.33}$$

The junction conditions in (8.126) and (8.127) matches the trace-free parts and the trace parts of the extrinsic curvature tensors for the two spacetime regions respectively, across the bounding surface. The expression for the extrinsic curvature tensor is given by

$$K_{ij} = -N_\sigma \left( \frac{\partial^2 \psi^\sigma}{\partial \xi^i \partial \xi^j} + \Gamma^\sigma_{\alpha\beta} \frac{\partial \psi^\alpha}{\partial \xi^i} \frac{\partial \psi^\beta}{\partial \xi^j} \right), \tag{7.34}$$

where $\xi^i \equiv (\tau, \theta, \phi)$ are the coordinates on the hypersurface, $\Gamma^\sigma_{\alpha\beta}$ are the Christoffel symbols for the spacetime whose extrinsic curvature components are being evaluated, $N_\sigma$ is the normal to the hypersurface, and $\psi^\sigma$ are the coordinates of the 4D-spacetime.

The trace of the extrinsic curvature tensor is given by

$$K = h^{ij} K_{ij}, \tag{7.35}$$

where $h_{ij}$ is the metric on the hypersurface. The traceless part of the extrinsic curvature tensor is given by

$$\tilde{K}_{ij} = K_{ij} - \frac{1}{3} h_{ij} K. \tag{7.36}$$

Matching the quantities $K$ and $\tilde{K}_{ij}$ across the boundary of the collapsing matter, we have

$$\left[ -K^-_{\tau\tau} + \frac{2}{\mathcal{R}} K^-_{\theta\theta} \right]_\Sigma = \left[ -K^+_{\tau\tau} + \frac{2}{\mathcal{R}} K^+_{\theta\theta} \right]_\Sigma, \tag{7.37}$$

$$\left[ K^-_{\tau\tau} + \frac{1}{\mathcal{R}^2} K^-_{\theta\theta} \right]_\Sigma = \left[ K^+_{\tau\tau} + \frac{1}{\mathcal{R}^2} K^+_{\theta\theta} \right]_\Sigma, \tag{7.38}$$

which when combined together, gives us the condition

$$\left( K^-_{ij} \right)_\Sigma = \left( K^+_{ij} \right)_\Sigma, \tag{7.39}$$

which is the same as the second Darmois-Israel junction condition [172, 173]. The condition (7.39) leads us to the following two equations :

$$-\left( \frac{A'}{AB} \right)_\Sigma = \left[ \left( \frac{dv}{d\tau} \right)^{-1} \left( \frac{d^2 v}{d\tau^2} \right) - \frac{1}{Y} \left( \frac{dv}{d\tau} \right) \left( \frac{M}{Y} - \frac{dM}{dY} \right) \right]_\Sigma, \tag{7.40}$$

$$\left( \frac{CC'}{B} \right)_\Sigma = \left[ Y \left( \frac{dv}{d\tau} \right) \left( \left( \frac{dv}{d\tau} \right)^{-2} - \left( \frac{dY}{d\tau} \right) \left( \frac{dv}{d\tau} \right)^{-1} \right) \right]_\Sigma. \tag{7.41}$$

The junction conditions (8.125), (8.126) and (8.127) deal with terms that are purely geometrical in their origin. In order to examine the rest of the junction conditions, which involve the continuity of the Ricci scalars, the traces, and their derivatives across the boundary, we now need to look into the nature of the energy-momentum tensor in the interior spacetime, since the Ricci scalar can be expressed in terms of the matter-energy components with the help of the field equations, and the trace originates from the energy-momentum tensor.

Using the junction conditions (8.125), (8.126) and (8.127), along with (7.18) and (7.21), and noting that $m(r,t)$ is the total mass-energy content inside the collapsing matter at a time $t$ and radius $r$, which is the same as $M(v,Y)$ in (8.95), we obtain

$$-\frac{C}{2}\left[(1+2\lambda)\left(p-q\right)-2\lambda L_m+\lambda T\right]|_{\Sigma}=\frac{1}{Y}\frac{dM}{dY}\;|_{\Sigma}\;. \tag{7.42}$$

However, $m$ and $M$ have different functional dependence, and $\frac{dm}{dr}$ and $\frac{dM}{dY}$ are not necessarily equal, because these variations are not similar in general. From equation (7.42), it is evident that the pressure is related to the heat flux.

The junction conditions (8.128) and (8.129) implies that

$$f(R,T)^{+}=f(R,T)^{-}. \tag{7.43}$$

Further, (8.128) also gives us

$$R^{-}\;|_{\Sigma}=R^{+}\;|_{\Sigma}, \tag{7.44}$$

which when written in terms of the interior and the exterior field equations, yields

$$(1+2\lambda)\left(-\rho_{int}+3p\right)-8\lambda L_m+2f^{-}\;|_{\Sigma}=(1+2\lambda)\left(2\rho+6P\right)-8\lambda L_{m_{ext}}+2f^{+}\;|_{\Sigma}, \tag{7.45}$$

while the condition (8.129) gives us

$$-\rho_{int}+3p=2\rho+6P. \tag{7.46}$$

Combining the four equations (7.43) to (7.46), we get

$$-8\lambda L_m\;|_{\Sigma}=-8\lambda L_{m_{ext}}\;|_{\Sigma}\;. \tag{7.47}$$

For $\lambda=0$, the $f(R,T)$ theory reduces to the case of Einstein's General Relativity. Hence, $\lambda$ must be non-vanishing for $f(R,T)$ theory. For the $f(R,T)$ junction conditions to be satisfied, the interior matter Lagrangian must be equal to the exterior matter Lagrangian at the boundary of the collapsing matter. Hence the

$f(R,T)$ junction conditions impose a restriction on the choice of the matter Lagrangian. This is evident since the matter-Lagrangians for both the interior and the exterior spacetimes were considered to be the same while the junction conditions were formulated in [230]. If there is minimal coupling between the collapsing matter and the gravity, the interior matter Lagrangian can be chosen either as the interior matter-energy density $-\rho_{int}$, or the isotropic pressure of the matter, i.e. $p$ [96, 54]. This freedom of choice is no longer available if the pressure of the interior matter is anisotropic. In this work, we have chosen the interior matter Lagrangian to be given by the interior energy density $\rho_{int}$.

The junction condition (8.130) then gives us

$$-\frac{\partial\rho_{int}}{\partial t}+3\frac{\partial p}{\partial t}\,|_{\Sigma}=2\frac{\partial\rho}{\partial v}+6\frac{\partial P}{\partial v}\,|_{\Sigma}, \tag{7.48}$$

$$-\frac{\partial\rho_{int}}{\partial r}+3\frac{\partial p}{\partial r}\,|_{\Sigma}=2\frac{\partial\rho}{\partial Y}+6\frac{\partial P}{\partial Y}\,|_{\Sigma}\,. \tag{7.49}$$

In order to examine the final junction condition (8.131), the Ricci scalars of the two spacetimes are expressed in terms of the respective field equations, so that (8.131) leads us to the two equations

$$(1+2\lambda)\left(-\frac{\partial\rho_{int}}{\partial t}+3\frac{\partial p}{\partial t}\right)-8\lambda\frac{\partial L_m}{\partial t}+2\frac{df(R,T)^-}{dt}\,|_{\Sigma}$$
$$=(1+2\lambda)\left(2\frac{\partial\rho}{\partial v}+6\frac{\partial P}{\partial v}\right)-8\lambda\frac{\partial L_{m_{ext}}}{\partial v}+2\frac{df(R,T)^+}{dv}\,|_{\Sigma}, \tag{7.50}$$

$$(1+2\lambda)\left(-\frac{\partial\rho_{int}}{\partial r}+3\frac{\partial p}{\partial r}\right)-8\lambda\frac{\partial L_m}{\partial r}+2\frac{df(R,T)^-}{dr}\,|_{\Sigma}$$
$$=(1+2\lambda)\left(2\frac{\partial\rho}{\partial Y}+6\frac{\partial P}{\partial Y}\right)-8\lambda\frac{\partial L_{m_{ext}}}{\partial Y}+2\frac{df(R,T)^+}{dY}\,|_{\Sigma}\,. \tag{7.51}$$

In view of the junction condition (8.130), we see that the bracketed terms in each of the above two equations cancel out on either sides. Using $f^- = R^- + 2\lambda T^-$ and $f^+ = R^+ + 2\lambda T^+$, we arrive at the following relation :

$$[\partial_\mu L_m]^+_- = 0. \tag{7.52}$$

Hence, like the Ricci scalar and the trace and their derivatives, the matter Lagrangian and its derivative for the two spacetimes also need to be continuously matched across the boundary.

## 7.6 Formation of Singularity

We now proceed to examine the nature of the end state of gravitational collapse. As the particles in the collapsing ball get closer to each another in course of time, eventually a state of extreme high density and curvature will be reached, where there is a breakdown of the normal laws of physics, leading to a 'singularity'. It is worthwhile to determine the time at which the singularity formation will take place, and whether the singularity will be a black hole, or a 'naked singularity' (which is visible to the external observer). For this purpose, the temporal dependence of the physical radius of the collapsing star have to be examined in detail.

In view of the choice for the interior matter Lagrangian as $-\rho_{int}$ , the interior field equations now reduce to

$$\frac{A''}{AB^2} - \frac{1}{A^2}\left(\frac{\ddot{B}}{B} + \frac{2\ddot{C}}{C}\right) + \frac{\dot{A}}{A^3}\left(\frac{\dot{B}}{B} + \frac{2\dot{C}}{C}\right) - \frac{A'B'}{AB^3} + \frac{2A'C'}{AB^2C}$$
$$= \rho_{int} - \frac{R}{2} - \lambda T, \tag{7.53}$$

$$-\frac{A''}{AB^2} - \frac{2C''}{B^2C} + \frac{\ddot{B}}{A^2B} - \frac{\dot{A}\dot{B}}{A^3B} + \frac{2\dot{B}\dot{C}}{A^2BC} + \frac{A'B'}{AB^3} + \frac{2B'C'}{B^3C}$$
$$= p + 2\lambda\left(p + \rho_{int}\right) + \frac{R}{2} + \lambda T, \tag{7.54}$$

$$\frac{1}{C^2}\left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2}\right) + \frac{C}{A^2}\left(\frac{\ddot{C}}{C^2} + \frac{\dot{C}\dot{B}}{C^2B} - \frac{\dot{C}\dot{A}}{C^2A}\right) - \frac{C''}{B^2C} + \frac{B'C'}{B^3C} - \frac{A'C'}{AB^2C}$$
$$= p + 2\lambda\left(p + \rho_{int}\right) + \frac{R}{2} + \lambda T, \tag{7.55}$$

$$\frac{\dot{C}'}{C} - \frac{A'\dot{C}}{AC} - \frac{\dot{B}C'}{BC} = (1 + 2\lambda)\frac{qAB}{2}. \tag{7.56}$$

On account of isotropy of pressure, equations (7.54) and (7.55) yield us

$$-\frac{1}{AB^2}\left(A''-\frac{A'B'}{B}-\frac{A'C'}{C}\right)+\frac{1}{A^2B}\left(\ddot{B}-\frac{\dot{A}\dot{B}}{A}+\frac{\dot{B}\dot{C}}{C}\right)-\frac{1}{B^2C}\left(C''-\frac{B'C'}{B}\right)$$
$$-\frac{1}{A^2C}\left(\ddot{C}-\frac{\dot{A}\dot{C}}{A}\right)-\frac{1}{C^2}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)=0. \tag{7.57}$$

Let us assume that the geometry of the dissipative collapse is described by the following profile [35]:

$$A(r,t)=A_0(r)s_1(t), \tag{7.58}$$
$$B(r,t)=B_0(r)s_2(t), \tag{7.59}$$
$$C(r,t)=rB_0(r)s_3(t). \tag{7.60}$$

Here, $A_0(r)$ and $B_0(r)$ represent the static perfect fluid configurations just before the collapse begins. With the onset of collapse at $t=0$, the collapsing ball behaves as a non-adiabatic fluid undergoing dissipation in the form of thermal radiation, and the dissipative effects are dependent on time. This time-dependence is encoded in the functions $s_1(t)$, $s_2(t)$ and $s_3(t)$, which, when subjected to appropriate rescaling of the coordinate time, as in [27, 28, 35], can be replaced by a single temporal function $w(t)$ associated with the cross-section of the collapsing fluid normal to the radial direction. A way of rescaling is to consider $s_1(t)=s_2(t)$, and $w(t)=\frac{s_3(t)}{s_1(t)}$, and make the transformations $s_i(t)\rightarrow\frac{s_i(t)}{s_1(t)}$ where $i=1,2,3$ . Metric coefficients separable in spatial and temporal parts, have been previously considered for dissipative collapse of charged cylindrical anisotropic fluid in GR by Guha and Banerji [35].

Therefore, we can write

$$A(r,t)=A_0(r), \tag{7.61}$$
$$B(r,t)=B_0(r), \tag{7.62}$$
$$C(r,t)=rB_0(r)w(t). \tag{7.63}$$

It may be mentioned here that the assumption of the same time dependence for all three metric coefficients leads us to a time independent equation of pressure isotropy, which cannot be used to find the form of the function $w(t)$. Hence such an assumption is not useful in this case.

Substituting the expressions for the metric coefficients defined in (7.61),(7.62) and (7.63) in the equation (7.57), we get

$$\frac{1}{A_0^2}\left(\frac{\ddot{w}}{w}+\frac{\dot{w}^2}{w^2}\right)+\frac{1}{r^2B_0^2w^2}-D(r)=0, \tag{7.64}$$

where

$$D(r)=-\frac{1}{A_0B_0^2}\left(A_0''-\frac{2A_0'B_0'}{B_0}-\frac{A_0'}{r}\right)-\frac{B_0''}{B_0^3}+\frac{B_0'}{rB_0^3}+\frac{2B_0'^2}{B_0^4}+\frac{1}{r^2B_0^2}. \tag{7.65}$$

The function $D(r)$ contains all the radial derivatives. From a single differential equation containing radial derivatives of both $A_0(r)$ and $B_0(r)$, it is not possible to find individual functional forms for either $A_0(r)$ or $B_0(r)$. However, since we are interested in determining the time of formation of singularity at the end of collapse, it is useful to examine the time evolution of the radius of the collapsing matter, for which we aim to solve this second order differential equation (7.64) for the temporal part $w(t)$ of the sectional radius $C(r,t)$. Once the form of $w(t)$ is determined, the time of formation of singularity can be obtained from its functional form. In that case, a non-vanishing $D(r)$ will yield a different form of $w(t)$ compared to the case when $D(r)=0$ .

We examine the cases for which $D(r)\neq 0$, and $D(r)=0$, separately.

### 7.6.1 Case I : $D(r)\neq 0$

This is the more complicated case and can only be analysed logically, as is outlined below.

We solve equation (7.64) for $w$ with the help of GRTensor software package, to obtain

$$w=\pm\frac{1}{\sqrt{2Dk^3}}\left(C_1Hck^2-2dk-C_2Hc\right)^{1/2}, \tag{7.66}$$

where $C_1$ and $C_2$ are integration constants, and

$$c(r)=\frac{1}{A_0^2}, \tag{7.67}$$

$$d(r)=\frac{1}{r^2B_0^2}, \tag{7.68}$$

$$H(r)=\sqrt{\frac{2D}{c}}, \tag{7.69}$$

$$k=\exp(Ht). \tag{7.70}$$

Since $w$ denotes the temporal dependence of the sectional radius which cannot be negative, we consider only the positive solution. For collapse to progress, we require $\dot{w} < 0$. A singularity will occur at the end state of collapse when the star will collapse to zero volume, as $w(t) \to 0$. This is where the singularity occurs at the final state of the collapse, which is different from the regular center of the star at $r = 0$ before the formation of singularity where the values of the physical parameters like the star density and the curvature scalars do not diverge [178].

The situation for which $w(t) = 0$, provides us the time of formation of the singularity $(t_s)$:

$$t_s = \frac{1}{H} \ln\left(\frac{d \pm \sqrt{d^2 + C_1 C_2 (Hc)^2}}{C_1 Hc}\right). \tag{7.71}$$

For real values of the discriminant in (7.71), we get the condition

$$-\frac{d^2}{(Hc)^2} \leq C_1 C_2, \tag{7.72}$$

and additionally, the argument of the logarithm needs to be positive.

As detailed in Appendix A, the parameters $H$ and $\frac{d}{c}$ must be constants. Therefore, $k$ is independent of $r$. These constants are now denoted by the following symbols :

$$H = C_3, \tag{7.73}$$

$$\frac{d}{c} = C_4. \tag{7.74}$$

If we examine equation (7.74) closely, we see that on account of (7.67) and (7.68), this ratio actually implies the condition

$$A_0^2 = C_4 r^2 B_0^2. \tag{7.75}$$

To understand the physical significance of this condition, we consider the radial null geodesics, which, for our interior line element are given by $ds^2 = 0$, with $\theta$ and $\phi$ as constant. In this case, using (7.75), we find (ignoring the constant),

$$\frac{dt}{dr} = \pm\frac{1}{r}, \tag{7.76}$$

or,

$$t = \pm ln\left|r\right|. \tag{7.77}$$

The slope of the curve tends to infinity as one approaches the regular center of the collapsing star at $r = 0$, and the light cones tend to close up, in the vicinity of that point. This means that a signal takes infinite time to reach an external observer from that point, and the singularity at that point is a black hole. The singularity at the center of the collapsing matter at $r = 0$ is not a removable singularity which can be avoided by a mere change of coordinates. It is the point at which the sectional radius $C(r,t)$ of the collapsing star becomes zero. The path of photons along the light cone boundary is also not a straight path, but a logarithmic curve. Hence our expression for the time of singularity formation now becomes

$$t_s = \frac{1}{C_3} \ln \left( \frac{C_4 \pm \sqrt{(C_4^2 + C_1 C_2 C_3^2)}}{C_1 C_3} \right), \tag{7.78}$$

which leads to the condition

$$-\frac{C_4^2}{C_3^2} \leq C_1 C_2, \tag{7.79}$$

for real values of the logarithm. Moreover, the logarithm needs to have a positive argument.

### 7.6.2 Case II : $D(r) = 0$

After putting $D(r) = 0$ in the differential equation (7.64) and solving it for $w$, with the help of GRTensor software package, while considering $w$ to be positive, since it is the temporal part of the sectional radius of the star, we obtain,

$$w = \left[ -\frac{d}{c} t^2 - 2C_5 t + 2C_6 \right]^{\frac{1}{2}}, \tag{7.80}$$

where $C_5$ and $C_6$ are integration constants. Since the ratio $\frac{d}{c}$ appears in the expression for $w(t)$, which has no radial dependence, we can use the condition $d/c = C_4$ in the homogeneous case also, at which, the above expression becomes,

$$w = \left[ -C_4 t^2 - 2C_5 t + 2C_6 \right]^{\frac{1}{2}}, \tag{7.81}$$

The significance of this ratio $\frac{d}{c}$ being a constant has been discussed under **"Case I : $D(r) \neq 0$"** of subsection A above. Solving for the time $t_s$ when $w = 0$, we get

$$t_s = \frac{-C_5 \pm \sqrt{y}}{C_4}, \tag{7.82}$$

where

$$y = C_5^2 + 2C_4C_6. \tag{7.83}$$

From this equation, for a real solution, the following constraint on the constants becomes evident immediately :

$$C_5^2 + 2C_4C_6 \geq 0. \tag{7.84}$$

The time derivative of $w$ is given by

$$\dot{w} = \frac{-C_4t - C_5}{w}. \tag{7.85}$$

To ensure collapse to progress, we require $\dot{w} < 0$. Hence at $t = t_s$, when $w \to 0$, we see that $\dot{w} \to -\infty$. We note that the right-hand side of the field equations for $R_{11}$ and $R_{22}$ involving $\lambda$ are identical, and get cancelled out as we apply the pressure isotropy condition. Hence the parameter $\lambda$ in the $f(R,T)$ function does not appear in the expression for $t_s$.

## 7.7 Formation of Apparent Horizon

The apparent horizon is a boundary between outward directed light rays which bend inwards, and those which move outwards. Several authors have determined the time of formation of apparent horizon for various types of collapse in GR [169, 170]. Chakrabarti and Banerjee [43] found apparent horizon formation time for a perfect fluid collapse in $f(R)$ gravity. Sharif and Kausar [36] found apparent horizons for spherically symmetric perfect fluid collapse in $f(R)$ theory. Amir and Sattar [51] found apparent horizons for the collapse of perfect fluid with spherical symmetry in $f(R,T)$ gravity. Abbas and Ahmed [53] studied apparent horizon formation for charged perfect fluid collapse in $f(R,T)$ theory. For the formation of the apparent horizon, we have the requirement that any outward normal on its boundary must be null. This condition can be expressed as

$$g^{\mu\nu}C_{,\mu}C_{,\nu} = 0, \tag{7.86}$$

which gives us the following condition:

$$\frac{\dot{C}^2}{A^2} = \frac{C'^2}{B^2}. \tag{7.87}$$

Using the expressions for $A$, $B$ and $C$ from (7.61), (7.62) and (7.63), we arrive at the relation

$$\frac{\dot{w}^2}{w^2} = \frac{\left(B_0 + rB_0'\right)^2 A_0^2}{r^2 B_0^4} = \delta^2. \tag{7.88}$$

Since, the left hand side is a function of $t$ only, and the right hand side is a function of $r$ only, so $\delta^2$ must be a constant. It is noted that $\delta = 0$ would imply that $w$ would become a constant and hence collapse would not be possible, since the value of $\delta$ should remain unchanged throughout the collapse. In order to examine the possible values of $\delta$, it is useful to solve the second equation of (7.88) first.

Using the previous expressions for $c$, $d$ and $C_4$ from (7.67), (7.68) and (7.74) in (7.88) which follows from the condition for apparent horizon formation, we can write

$$\frac{\delta^2}{C_4} = \frac{\left(B_0 + rB_0'\right)^2}{B_0^2}. \tag{7.89}$$

Integrating with respect to $r$, we get

$$B_0 = C_7 r^n. \tag{7.90}$$

Putting it back in the expression for $\delta^2$, we obtain

$$A_0 = \frac{C_7 r^{n+1}\delta}{n+1}. \tag{7.91}$$

Here, $C_7$ is the integration constant, and $n$ is a real number not equal to -1, so as to keep the metric coefficient $A_0$ finite. We also see that $D(r)$ does not vanish with these expressions for $A_0$ and $B_0$. Hence, these power series solutions are valid only for the case when $D(r) \neq 0$. It follows from the condition for apparent horizon, and the form of the metric coefficients, that apparent horizon will be formed only in the case when $D(r) \neq 0$.

For collapse to proceed, it is necessary that the condition $\dot{w} < 0$ be satisfied. Using the form of $w$ and $\dot{w}$ obtained from equation (7.66), where $D(r) \neq 0$, this leads us to the following constraint :

$$e^{-2\delta t}\left(\delta + 6C_2 e^{-2\delta t}\right) - 2C_1 < 0. \tag{7.92}$$

Using the power series forms of $A_0(r)$ and $B_0(r)$ from equations (7.91) and (7.90), we find that

$$C_3 = H = 2\delta, \tag{7.93}$$

and,

$$C_4 = \frac{A_0^2}{r^2 B_0^2} = \frac{\delta^2}{(n+1)^2}, \tag{7.94}$$

which is a positive quantity for real values of $\delta$ and $n$. We choose $n = 1$ here onwards for our purpose.

The expression for the time of formation of singularity given by equation (7.78) is independent of the radial coordinate $r$. So it is expected that all the shells of the collapsing spherical matter ball will collapse to the final singularity at the same time $t_s$. In such a case, the physical radius of the outer shells should decrease at a faster rate compared to the inner shells, without however crossing each other (thus avoiding any shell-crossing), so that all the shells reach the final singularity at the same time.

Equation (7.78) can be rewritten in the form

$$t_s = \frac{1}{2\delta} \ln\left(\frac{\delta^2 \pm 4\sqrt{z}}{8\delta C_1}\right). \tag{7.95}$$

where,

$$z = C_4^2 + C_1 C_2 C_3^2 = \frac{\delta^4}{16} + 4\delta^2 C_1 C_2. \tag{7.96}$$

For real values of the discriminant, $z$ must be positive definite. Using the forms of $w$ and $\dot{w}$ obtained from equation (7.66) where $D(r) \neq 0$, and using them in the condition for apparent horizon, that is,

$$\dot{w}^2 = \delta^2 w^2, \tag{7.97}$$

and solving for the apparent horizon formation time, and utilising equation (7.78), we have

$$t_{ah_1} - t_s = \frac{1}{2\delta} \ln\left(\frac{-64 C_1 C_2}{\delta\left(\delta \pm \sqrt{\delta^2 + 64 C_1 C_2}\right)}\right), \tag{7.98}$$

and,

$$t_{ah_2} - t_s = \frac{1}{2\delta} \ln\left(\frac{3\delta \pm \sqrt{9\delta^2 + 512 C_1 C_2}}{2\delta \pm \sqrt{4\delta^2 + 256 C_1 C_2}}\right). \tag{7.99}$$

When expressed in terms of $z$ and $\delta$, we have,

$$t_{ah_1} - t_s = \frac{1}{2\delta} \ln\left(\frac{\delta^4 - 16z}{\delta^2\left(\delta^2 \pm 4\sqrt{z}\right)}\right) \tag{7.100}$$

and,

$$t_{ah_2} - t_s = \frac{1}{2\delta} \ln \left( \frac{3\delta^2 \pm \sqrt{\delta^4 + 128z}}{2\left(\delta^2 \pm 4\sqrt{z}\right)} \right) \tag{7.101}$$

## Conditions for Formation of Black Hole

Joshi, Goswami and Dadhich [168] showed that in a case like above, where the time of formation of singularity is not dependent on the radial coordinate, no naked singularity can result. So it is necessary to examine the constraints on the parameters $\delta$, $C_1$ and $C_2$ for black hole formation to be possible.

If a black hole is to be formed, it is necessary that $t_{ah} - t_s < 0$. This would imply that $t_{ah} < t_s$, which shows that the apparent horizon forms before the singularity. In other words, the singularity is hidden behind the apparent horizon, and is a black hole.

Let us examine the constraints on the possible cases for this inequality to hold.

## Case I : the condition $z > 0$

1. For the expression $t_{ah_1} - t_s$ from equation (7.100) to be negative for black hole formation, the following constraints are to be obeyed :

Table 7.1: Constraints on $C_1$ and $C_2$ from $t_{ah_1} - t_s < 0$ for Black Hole Formation

| | **Sign of $\delta$** | **Sign chosen from $\pm$ from denominator** | **Condition for Black Hole Formation** |
|---|---|---|---|
| **1.** | $\delta > 0$ | $-$ | Black Hole formation is not possible |
| **2.** | $\delta > 0$ | $+$ | $-(\delta^2/64) < C_1C_2 < 0$ |
| **3.** | $\delta < 0$ | $+$ | $-(\delta^2/64) < C_1C_2 < 0$ or $C_1C_2 > 0$ |
| **4.** | $\delta < 0$ | $-$ | Black Hole formation is not possible |

2. For the expression $t_{ah_2} - t_s$ from equation (7.101) to be negative for black hole formation, the following constraints are to be obeyed :

Table 7.2: Constraints on $C_1$ and $C_2$ from $t_{ah_2} - t_s < 0$ for Black Hole Formation

| | **Sign** of $\delta$ | **Sign chosen from $\pm$ in numerator** | **Sign chosen from $\pm$ in denominator** | **Condition for Black Hole Formation** |
|---|---|---|---|---|
| **1.** | $\delta > 0$ | $+$ | $+$ | Black Hole formation is not possible |
| **2.** | $\delta > 0$ | $-$ | $-$ | Black Hole formation is not possible |
| **3.** | $\delta > 0$ | $+$ | $-$ | Black Hole formation is not possible |
| **4.** | $\delta > 0$ | $-$ | $+$ | $C_1C_2 > -(\delta^2/64)$ |
| **5.** | $\delta < 0$ | $+$ | $+$ | $-(\delta^2/64) < C_1C_2 < 0$, or $C_1C_2 > 0$ |
| **6.** | $\delta < 0$ | $-$ | $-$ | $-(\delta^2/64) < C_1C_2$ |
| **7.** | $\delta < 0$ | $+$ | $-$ | Black Hole formation is not possible |
| **8.** | $\delta < 0$ | $-$ | $+$ | $-(\delta^2/64) < C_1C_2 < 0$ |

## Case II : the condition $z = 0$

1. For $z = 0$, the expression $t_{ah_1} - t_s$ from equation (7.100) becomes zero, and black hole formation is not possible.

2. For the expression $t_{ah_2} - t_s$ from equation (7.101) to be negative for black hole formation, the following constraints are to be obeyed :

Table 7.3: Conditions for Black Hole Formation from $t_{ah_2} - t_s < 0$

| | **Value of $t_{ah_2} - t_s$** | **Condition for Black Hole Formation** |
|---|---|---|
| **1.** | 0 | Black Hole formation is not possible |
| **2.** | $\frac{1}{2\delta} \ln 2$ | $\delta < 0$ |

3. There is an additional constraint for the case $z = 0$ : $\delta$ **and** $C_1$ **should have the same sign**.

## 7.8 Summary and Discussions

To summarize, we have started with a generalized Vaidya exterior metric, considering the exterior spacetime to the collapsing matter to be filled with a combination of Type-I and Type-II matter fluid. We investigate the non-adiabatic collapse of an isotropic matter ball involving heat flux. The form of the temporal dependence of the physical radius from the pressure isotropy condition by utilising the field equations in $f(R,T)$ theory of gravity for a collapsing matter involving heat flux had not been previously found in any literature. Proceeding with the collapse formalism, the field equations are formed, the $f(R,T)$ junction conditions are applied with the generalised Vaidya exterior solution. Finally, for determining the time of formation of the final singularity at the end of the collapse, the metric coefficients for the interior spacetime, $A(r,t)$ and $B(r,t)$ are assumed to be functions of the radial coordinate, and, in case of $C(r,t)$ which represents the physical radius of the collapsing fluid, to be separable in the spatial and temporal coordinates. Using these forms of the metric parameters, a differential equation in the temporal function $w$

is obtained by utilising the condition of pressure isotropy. By requiring $w(t) = 0$, the time of the singularity formation $t_s$ is obtained with the necessary restrictions imposed on the integration constants, for two cases : i) when $D(r) \neq 0$, and ii) when $D(r) = 0$. A power series solution in the radial coordinates is obtained for $A_0(r)$ and $B_0(r)$, from the condition for apparent horizon formation and it is seen that $D(r)$ is non-vanishing in that case. The time of formation of apparent horizon, $t_{ah}$ is also obtained. The difference between $t_{ah}$ and $t_s$ for $D(r) \neq 0$ is calculated, and the constraints on the parameters $\delta$, $C_1$ and $C_2$ are examined for the final singularity to be a black hole.

The following observations can be made in the context of this work :

1. We have shown in our analysis, just as Rosa had considered a priori in [230] while formulating the $f(R,T)$ junction conditions, that the interior matter Lagrangian must match the exterior matter Lagrangian at the boundary of the collapsing matter. Also, the derivatives of the matter Lagrangian with respect to the coordinates of their respective spacetimes, should be continuous across the boundary. It is worth noting that this condition arises from the matching of the Ricci scalars and the traces of the energy momentum tensors of the interior and the exterior regions across the boundary of the collapsing matter, which are the junction conditions for $f(R,T)$ theory. These conditions need not be satisfied in GR, as a result of which it is not necessary in GR for the matter Lagrangians and their derivatives to be matched at the boundary. It is seen that the absence of heat flux (perfect fluid case), and utilising equations (7.61), (7.62) and (7.63) would give us a relation between $A_0(r)$ and $B_0(r)$ from equation (7.21). Their ratio would still be linear in the radial coordinate.

2. If the pressure was anisotropic, then the pressure in equation (7.42) would be the radial pressure, since the field equation for $R_{11}$, which is the radial component of the Ricci tensor, was used to arrive at it.

3. In case of pressure anisotropy, we would need to find a different way to arrive at the solution for the temporal function $w$. In that case, equating the expression for $q$ from equation (7.42) and equation (7.21) and using the equations (7.61), (7.62) and (7.63) would have provided a differential equation for $w$ with additional terms involving $w$ and its time derivative. This would have resulted in a more complicated relation to work with.

4. Presence of shear would have further added terms involving $w$ and its time derivative to the differential equation, since it would have appeared in the field

equation for $R_{11}$ and consequently in equation (7.42). Hence the corresponding differential equation would become more complicated if effects like anisotropy and shear are involved in the collapse.

5. The parameter $\lambda$ has no effect on either the singularity formation or the apparent horizon formation : the former because it gets cancelled out as the pressure isotropy condition is applied, and the latter because the condition for the formation of apparent horizon remains the same in $f(R,T)$ theory, as in GR.

6. We see that for a black hole to form as the final state of the collapse, $\delta$, $C_1$ and $C_2$ get constrained in different ways for the possible expressions of $t_{ah} - t_s$, obtained from the condition for apparent horizon, for non-vanishing $D(r)$, both when $z$ is zero and non-zero.

7. Tables I and II enlist the constraints that must be satisfied for a black hole to form when $z > 0$ for two possible expressions of $t_{ah} - t_s$, while Table III shows the constraints required for black hole formation for the $z = 0$ case.

8. The chosen configuration of the collapsing matter does not involve shear viscosity. It has been reasoned both in GR [179], as well as in $f(R,T)$ theory [180] that when shearing effects come into play, the matter concentration will be resisted by these forces. As a result, formation of a trapped surface or apparent horizon will be delayed, and the probability of a naked singularity as the final state of the collapse will be greater. It is also concluded in [179], that the absence of shear would necessarily have a black hole as the final state of collapse. Since the retardation of the collapse and delay of formation of apparent horizon is a result of the shearing effect and not specifically dependent on the gravity theory chosen, it is reasonable to predict that in our case, where the collapse takes place in $f(R,T)$ gravity, and in absence of shear, the end result must necessarily be a black hole. Hence the constraints obtained on the parameters $\delta$, $C_1$ and $C_2$, will have to be adhered to.

9. In this paper, we have considered the gravitational collapse of a spherical non-rotating body where a spherical black hole will be formed as the final state of the collapse in the cases where black hole formation is possible [181]. Since the collapsing matter is radiating heat in the radial direction, mass-energy content of the collapsing body will decrease with time. In a previous paper of ours [54], equation (40) provides an expression for the temporal variation of the mass-energy content of a collapsing matter in $f(R,T)$ gravity. Adjustments can be

made in that expression which would fit our case for an $f(R,T)$ function linear in both $R$ and $T$, pressure isotropy and absence of shear and free-streaming radiation. The adjusted equation reads

$$D_T M = -\frac{C^2}{2f_R}\left[U\left\{(1+f_T)p + \rho f_T + \frac{1}{2}(f - Rf_R)\right\} + H(1+f_T)q\right]. \tag{7.102}$$

All the terms of the expression are explained in [54]. Since the time of formation of apparent horizon has been calculated, the amount of mass-energy content left in the collapsing sphere just after the apparent horizon forms, may be determined via an integration over time from 0 to $t_{ah}$ provided the functional forms of all the terms are explicitly known. This may give us an indication of the mass of the black hole formed. As stated in [171, 182], in the case of spherical collapse of uncharged matter, the event horizon and apparent horizon coincide as the collapsed matter reaches the final static state. Hence the same will be true in our case.

In this context, we want to mention that instead of considering a linear function of $R$ and $T$, if we consider some other functional dependence of $R$, for example, a generalised function like $f(R,T) = \alpha R^n + \beta T$ where $\alpha$ and $\beta$ are constants, terms involving the dark source terms $D_{\mu\nu}$ and $f_R$ will appear in the pressure isotropy condition (7.57) and the resulting differential equation (7.64) will become still more complicated and difficult to solve. These terms would arise entirely from the geometry of the spacetime and the modified gravity function. It is only because we have chosen a linear $f(R,T)$ function, that $f_R$ becomes unity and all the components of $D_{\mu\nu}$ vanish. Consequently, the pressure isotropy condition gives rise to a relatively less complicated differential equation. The contribution from modified gravity will be present in equation (7.64) for other functional forms of $f(R,T)$. Also, equation (7.102), which describes the temporal variation of the mass-energy content of the collapsing matter, already has the terms $f_R$ and $f_T$ which comes from the modified gravity. Although $f_R$ becomes unity in our case, $f_T$ takes the value of the parameter $\lambda$. In the manner of the prescription provided in [183], extending the method to $f(R,T)$ gravity, black hole solutions may be obtained from the $f(R,T)$ field equations of the generalized Vaidya exterior spacetime, by integrating over the radial coordinate from the black hole horizon to the asymptotic region. Determining the black hole area from the horizon radius and consequently the surface gravity may be possible following the ideas about black hole thermodynamics in modified gravity [184, 185].

There is scope of further investigation as to what happens in presence of shear viscosity and pressure anisotropy in the collapsing star, which may be taken up in future.

# Appendix A

This is the elaborate method of obtaining equations (7.73) and (7.74), by showing why the ratio $\frac{d(r)}{c(r)}$ is a constant both for the case when $D(r) \neq 0$ and when $D(r) = 0$. In each case we start with the solution for the function $w(t)$.

For $D(r) \neq 0$, the solution of $w(t)$ is given by (7.66):

$$w(t) = \pm \frac{1}{\sqrt{2Dk^3}} \left( C_1 Hck^2 - 2dk - C_2 Hc \right)^{1/2}, \tag{7.103}$$

where $C_1$ and $C_2$ are integration constants,

$$c(r) = \frac{1}{A_0^2}, \tag{7.104}$$

$$d(r) = \frac{1}{r^2 B_0^2}, \tag{7.105}$$

$$H(r) = \sqrt{\frac{2D}{c}}, \tag{7.106}$$

and,

$$k = \exp(Ht). \tag{7.107}$$

From (7.103) and using (7.106), we get,

$$w^2 = \frac{C_1}{Hk} - \frac{d}{Dk^2} - \frac{C_2}{Hk^3}. \tag{7.108}$$

Since the left-hand side of the above equation is a function of $t$ only, the right-hand side must also be a function of $t$ and have no dependence on $r$. There are 3 additive terms on the right hand side. So each of them must individually be a function of $t$. Examining the first term on the right-hand side of (7.108), we see that $C_1$ must be a constant. So, $Hk$ must be independent of $r$. In other words, by requiring its derivative with respect to $r$ to vanish, and using (7.107), we have

$$H' \left(1 + Ht\right) e^{Ht} = 0. \tag{7.109}$$

where prime denotes derivative with respect to $r$. Now, $e^{Ht} \neq 0$ for real values of $H(r)$ and $t$. If we consider $1 + Ht = 0$, it implies,

$$H = -\frac{1}{t}, \tag{7.110}$$

which is a contradiction, since, $H$ is a function of $r$. So the only possibility for (7.109) to be satisfied, is that

$$H' = 0, \tag{7.111}$$

which implies that $H$ must be a constant, and thus equation (7.73) is obtained. From (7.106), we can also see that the ratio

$$\frac{D(r)}{c(r)} = \text{constant} = m_1. \tag{7.112}$$

Since $H$ is a constant, we also have from (7.107) that $k$ is a function of $t$ only. Taking the second term of equation (7.108), since $k = k(t)$, it follows that $d/D$ should also be a function of $t$ alone. But since, $d$ and $D$ are functions of $r$ without any temporal dependence, the only way this is possible is if their ratio is a constant. Hence we have,

$$\frac{d(r)}{D(r)} = \text{constant} = m_2. \tag{7.113}$$

Multiplying (7.112) and (7.113) and using the fact that $D(r) \neq 0$, we see that

$$\frac{d(r)}{c(r)} = m_1 m_2 = \text{constant} = C_4. \tag{7.114}$$

Thus equation (7.74) is obtained.

**Comments:**
For $D(r) = 0$, the solution of $w(t)$ is given by

$$w = \left[-\frac{d}{c}t^2 - 2C_5 t + 2C_6\right]^{\frac{1}{2}}, \tag{7.115}$$

where $C_5$ and $C_6$ are integration constants, $c$ and $d$ are as defined in (7.67) and (7.68). Squaring both sides,

$$w^2 = -\frac{d}{c}t^2 - 2C_5 t + 2C_6. \tag{7.116}$$

Since the left-hand side of (7.116) is a function of $t$ alone, the right-hand side must also be a function of $t$ and independent of $r$. It is obvious from the fact that $C_5$ and $C_6$ are constants, that the second and the third terms are function of $t$ only, and a constant, respectively. For the term $\frac{d}{c}t^2$ to be independent of $r$, the derivative of $\frac{d}{c}$ with respect to $r$ must vanish. But since both $d(r)$ and $c(r)$ are functions of $r$, the only way the radial derivative of this ratio would vanish is if $\frac{d}{c}$ is a constant. Hence, for both vanishing and non-vanishing $D(r)$, the ratio $\frac{d(r)}{c(r)}$ must be a constant.

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# Chapter 8

# Structure Scalars and Complexity Conditions

This chapter deals with the influence of structure scalars on the physical properties of a charged dissipative spherically symmetric collapsing matter in $f(R,T)$ gravity. The publication details of the material in this chapter are given below.

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"

## 8.1 Introduction

Investigations on the determination of the final fate of a stellar object collapsing under its own gravity gained prominence among researchers ever since the seminal work on the collapse of a spherical dust ball by Oppenheimer and Snyder [16], and independently by Datt [17] during 1938-39. The general relativistic consideration of a matter combination represented by a rank two energy-momentum tensor left open a number of possibilities for the end result of gravitational collapse, depending on the type of collapsing matter. Details of the involvement or absence of shear viscosity, pressure anisotropy, electric field, dissipative effects like heat flow, play a huge role in deciding whether the final fate of collapse will be a black hole or a naked singularity, a regime in which the usual laws of physics cannot be applied any more. Over the years, researchers have studied a number of such cases in detail in the context of General Relativity (GR) [18, 19, 58, 63, 20, 23, 24, 25, 26, 28, 29, 64, **?**].

In 2009, Herrera et al [58] demonstrated for the first time that a set of scalars derived from the orthogonal splitting of the Riemann tensor have distinct physical interpretation and are particularly suited for the characterisation of self-gravitating relativistic fluids. They termed these scalars as "structure scalars", which influence the various physical parameters of the collapsing matter. Their construction followed the consideration of Bel [56], who introduced for the first time the idea of the orthogonal splitting of the Riemann tensor by defining the dual tensors corresponding to the Riemann tensor, and led to the formulation of the gravitational super energy tensor. In 2008, Gómez-Lobo [64] in an effort to shed light on the true physical meaning of superenergy, performed the orthogonal splitting of the Bel and Bel-Robinson tensors, and analysed the different parts arising out of the splitting. A detailed presentation of the orthogonal splitting of the Riemann tensor was also provided in this paper.

Following the work of Herrera et al [58], several authors have characterized the evolution of self-gravitating systems in terms of structure scalars, in the context of General Relativity and also in other modified theories of gravity. Herrera, Di Prisco and Ibanez investigated the role of electric charge and the cosmological constant in the structure scalars [204]. Herrera, Di Prisco and Ospino studied structure scalars for relativistic fluids with cylindrical symmetry [205]. Sharif and Bhatti studied structure scalars for charged relativistic fluids in cylindrical symmetry [206]. They also studied the role of structure scalars in presence of charge in plane symmetry [207], charged static solutions with axial symmetry [190], structure scalars for tilted Szekeres geometry and also the super-Poynting vector [209]. Sharif and Manzoor analysed the role of structure scalars and obtained the inhomogeneity factors for the spherical self-gravitating fluid models in Brans-Dicke theory, and investigated the spherical static anisotropic solutions with inhomogeneity using these scalars [210]. They also studied the role of structure scalars in cylindrical systems in Brans-Dicke gravity, and showed that cylindrical systems must necessarily be inhomogeneous [211].

The fact that the universe is undergoing a late-time accelerated expansion emerged from the analysis of the data obtained from the observation of the Type Ia spernova

[3, 4]. Explanation for this accelerated expansion is provided by assuming the presence of a mysterious component called "dark energy", about which we do not have much information, except that it constitutes a large part of the total matter-energy content of the universe. Further, the galactic rotation curves show the presence of "dark matter" which forms the major constituent of the matter component of the universe. A modification of the Einstein-Hilbert action in GR, by replacing the Ricci scalar $R$ with a function of the Ricci scalar $f(R)$, yields us the $f(R)$ theory of gravity. This modification gives rise to some extra terms related to the curvature of the spacetime, which are purely geometrical in origin. These terms can be considered to explain the accelerated expansion of the universe, without the need to invoke the cosmological constant $\Lambda$, which is sometimes used in the Einstein Field Equations in GR for the same purpose. The works by De Felice and Tsujikawa [2], Sotiriou and Faraoni [7], and Capozziello and Laurentis [8] provide further insights into modified gravity theories. Modified gravity theories allowing a unification of early time inflation and late-time acceleration was also discussed in [70]. Development of modified gravity theories in the context of inflation, bounce and late time acceleration was discussed in [212]. A comparative study of gravitational collapse in General Relativity and $f(R)$ gravity, or more precisely, $R^2$ gravity can be found in [213]. Sharif and Yousaf studied structure scalars in the case of radiating cylindrical collapse in $f(R)$ gravity [214]. Bhatti, Yousaf and Tariq studied structure scalars and their evolution for massive bodies in $f(R)$ gravity [215], and analysed structure scalars in $f(R)$ gravity also in the presence of electric charge [216]. Further, they examined the role of structure scalars in the evolution of compact objects in Palatini $f(R)$ gravity [217].

An extension of the $f(R)$ theory is the $f(R,T)$ theory of gravity formulated by Harko et al [9]. In this theory the Ricci scalar $R$ in the Enstein-Hilbert action is replaced by a function of both $R$ and $T$, where $T$ is the trace of the energy-momentum tensor. Thus the $f(R,T)$ function depends both on the curvature of the spacetime, as well as the matter part. The inclusion of the trace takes into account the presence of imperfect exotic fluids or possible quantum effects such as the conformal anomaly. The extra curvature terms arising from the geometry can explain the accelerated expansion of the universe, without the need to invoke dark energy. In addition, the galactic rotation curves can also be explained without invoking dark matter. A suitable choice of the $f(R,T)$ function is a function which is linear in both $R$ and $T$. This linear form $R+\lambda T$ leads to power-law type of scale factors. Sahoo and his collaborators showed that this model of $f(R)+\lambda T$ gravity can be used as an alternative to the cosmic acceleration [155]. Guha and Ghosh studied dynamical conditions and causal transport phenomena for a dissipative spherical collapse in $f(R,T)$ gravity [54]. They also studied the formation of singularity and apparent horizon for dissipative spherical collapse in $f(R,T)$ gravity [198]. Yousaf et al [218] explored the evolution of compact objects in $f(R,T)$ gravity with the help of structure scalars for the case of spherical systems coupled with heat- and radiation-emitting shearing viscous matter distributions. They explicitly demonstrated that even in modified gravity, the evolutionary phases of relativistic stellar systems could be analyzed with the help of these scalar functions. In a separate paper [219] these authors examined irregularity factors for a self-gravitating spherical star evolving in the presence of an imperfect fluid in $f(R,T)$ gravity, where they demonstrated that, as the complexity of the matter with the anisotropic stresses increases, the inhomogeneity factor corresponds more closely to one of the structure scalars. Hussain et. al. studied the role of structure scalars in the $f(R,T)$ theory of gravity [220]. Yousaf, Bhatti and Farwa studied axial and reflection-symmetric self-gravitating systems and structure scalars in $f(R,T)$ gravity [221].

Any physical property of a collapsing system, apart from isotropic pressure and homogeneous energy density, adds to the complexity of the system. More precisely, parameters like energy density inhomogeneity, pressure anisotropy, dissipative and shearing effects, electromagnetic field, all add to the complexity of a collapsing fluid configuration, while the simplest system of a fluid with homogeneous energy density and isotropic pressure is said to possess a vanishing complexity. This new definition of complexity factor was introduced by Herrera [222] for static spherical self-gravitating systems, based on a quantity that appears in the orthogonal splitting of the Riemann tensor, in the context of general relativity. This proposal was immediately followed up by Herrera et al [223], who extended it to fully dynamic situation, where they also considered the condition of minimal complexity of the pattern of evolution. This treatment was applied further to other cases in the context of GR [224, 225]. Sharif and collaborators [226, 227] also examined the complexity factor for dynamical spherical systems under various conditions. Complexity factor have also been discussed in $f(R,T,R_{\mu\nu}T^{\mu\nu})$ gravity [228].

In this paper, we have chosen to work in the framework of $f(R,T)$ gravity. The arbitrariness in the choice of the $f(R,T)$ function leads to flexibility in the choice of models. It may lead to insights regarding the behaviour of gravity in high-energy scenarios by the introduction of quantum corrections in some of its functional forms. It may be used to model compact objects like black holes and neutron stars, and investigate their various properties including stability. It can also offer explanations to the formation of large-scale structures in the universe. The paper is divided into the following sections : In section II, a brief formalism of the $f(R,T)$ theory of gravity has been provided, followed by the definition of the structure scalars in section III. In section IV, the interior spacetime and the enrgy momentum tensor have been specified, and the structure scalars, the Weyl scalar, the expansion scalar and the shear scalar have been obtained in terms of the metric coefficients. The field equations have been presented in section V, and the relation between the physical matter variables and the structure scalars have been established in section VI, with subsection A discussing the constant $R$ and $T$ case for a dust ball, and subsection B showing the form of these relations for a linear $f(R,T)$ function, followed by a discussion on the complexity factor in subsection C. The exterior spacetime and the corresponding field equations have been presented in section VII, and the interior spacetime and the energy-momentum tensor in terms of the Ricci tensor in section VIII. The junction conditions in $f(R,T)$ gravity have been discussed in section IX. The energy conditions in $f(R,T)$ gravity have been discussed in section X. Finally, the results and conclusions are presented in section XI.

## 8.2 The $f(R,T)$ Formalism

The modified Einstein-Hilbert action in $f(R,T)$ gravity, is given by

$$S = \int d^4x\sqrt{-g}\left(\frac{f(R,T)}{16\pi G} + \mathcal{L}_m\right). \tag{8.1}$$

The $f(R,T)$ field equations are obtained by variation of this action, and are given by

$$G_{\mu\nu} = \frac{1}{f_R}\left[(1+f_T)\,T^{(m)}_{\mu\nu} - L_m g_{\mu\nu} f_T + \frac{1}{2}\left(f - Rf_R\right)g_{\mu\nu} - D_{\mu\nu}\right], \tag{8.2}$$

where $g_{\mu\nu}$ is the metric tensor representing the four-dimensional spacetime in the region interior to the collapsing matter, $G_{\mu\nu}$ is the Einstein tensor, $R$ is the Ricci scalar, $T$ is the trace of the energy-momentum tensor, $f_R$ and $f_T$ are the derivatives of the $f(R,T)$ function with respect to $R$ and $T$, respectively, $L_m$ is the interior matter Lagrangian, and $D_{\mu\nu} = \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) f_R$ which includes the higher order curvature terms, which acts as the source of dark energy. Before describing the interior spacetime for our investigation of the collapse, we formulate the structure scalars, which play an important role in influencing the physical parameters of the collapsing matter.

## 8.3 Structure Scalars

Following Bel [56] and Herrera [58], we formulate the quantities termed "structure scalars" by an orthogonal splitting of the Riemann tensor $R_{\alpha\beta\gamma\delta}$ for the interior spacetime. A prescription for this orthogonal splitting was given by Gomez-Lobo [64]. In [58], Herrera showed that in the framework of GR, these scalar quantities can be utilised to completely describe the structure and evolution of the spherically symmetric anisotropic dissipative fluid which is self-gravitating. These scalars are found to influence the physical properties of the fluid, such as, the energy density, pressure anisotropy, heat flux, and the active gravitational mass, and combinations of these scalars are sufficient to express the solutions to the Einstein Field Equations in the static case.

In analogy with Herrera [58], the right dual, left dual, and double dual of the Riemann tensor are specified as follows :

$$R^*_{\alpha\beta\gamma\delta} = \frac{1}{2}\eta_{\epsilon\rho\gamma\delta}R^{\epsilon\rho}{}_{\alpha\beta} \tag{8.3}$$

$$^*R_{\alpha\beta\gamma\delta} = \frac{1}{2}\eta_{\alpha\beta\epsilon\rho}R^{\epsilon\rho}{}_{\gamma\delta} \tag{8.4}$$

$$^*R^*_{\alpha\beta\gamma\delta} = \frac{1}{2}\eta_{\alpha\beta}{}^{\epsilon\rho}R^*_{\epsilon\rho\gamma\delta} \tag{8.5}$$

where, $\eta_{\alpha\beta\gamma\delta}$ is the four-index Kronecker delta symbol.

The spacetime metric is described by the tensor $g_{\mu\nu}$ and the four-velocity vector is given by $u^\mu$. Utilising equations (8.3), (8.4) and (8.5), and following the prescription by Herrera [58], we have :

$$Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta}u^\gamma u^\delta \tag{8.6}$$

$$X_{\alpha\beta} = {}^* R^*_{\alpha\gamma\beta\delta}u^\gamma u^\delta = \frac{1}{2}\eta^{\epsilon\rho}{}_{\alpha\gamma}R^*_{\epsilon\rho\beta\delta}u^\gamma u^\delta \tag{8.7}$$

$$Z_{\alpha\beta} = {}^* R_{\alpha\gamma\beta\delta}u^\gamma u^\delta = \frac{1}{2}\eta_{\alpha\gamma\epsilon\rho}R^{\epsilon\rho}{}_{\beta\delta}u^\gamma u^\delta \tag{8.8}$$

Denoting the trace part and trace-free part of $X_{\alpha\beta}$ by $X_T$ and $X_{TF}$ respectively, and the trace part and trace-free part of $Y_{\alpha\beta}$ by $Y_T$ and $Y_{TF}$ respectively, with $\chi_\mu$ as a unit vector in the radial direction, we can write, using the projection tensor $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$,

$$X_{\alpha\beta} = \frac{1}{3}X_T h_{\alpha\beta} + X_{TF}\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right) \tag{8.9}$$

and,

$$Y_{\alpha\beta} = \frac{1}{3}Y_T h_{\alpha\beta} + Y_{TF}\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right) \tag{8.10}$$

The trace parts and trace-free parts of these vectors are referred to as structure scalars since it can be shown from the field equations for the interior spacetime that the structure scalars influence various matter-energy parameters like the inhomogeneity in the matter-energy density, shear viscosity, pressure anisotropy, electric field and so on.

## 8.4 The Interior Spacetime and Energy-Momentum Tensor

The metric describing the interior spacetime is the most general spherically symmetric metric which is

$$ds_-^2 = -A(r,t)^2dt^2 + B(r,t)^2dr^2 + C(r,t)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{8.11}$$

The matter in the interior spacetime is considered to be an anisotropic fluid undergoing dissipation in the form of heat flux and free-streaming radiation, with shear viscosity, whose energy-momentum tensor is given by

$$T_{\mu\nu} = (\rho + p_\perp)\,u_\mu u_\nu + p_\perp g_{\mu\nu} + (p_r - p_\perp)\,\chi_\mu\chi_\nu + q_{(\mu}u_{\nu)} + \epsilon l_\mu l_\nu - 2\eta\sigma_{\mu\nu} \tag{8.12}$$

where, $\rho$ is the interior energy density, $p_\perp$ is the tangential pressure, $p_r$ is the radial pressure, $u^\mu$ is the four-velocity, $\chi^\mu$ is a unit vector in the radial direction, $q_\mu$ is the heat flux vector in the radial direction and is given by $q_\mu = q\chi_\mu$, with $q$ being the heat flux, $\epsilon$ is the free-streaming radiation density, $l^\mu$ is a null 4-vector, $\eta$ is the coefficient of shear viscosity, which is positive, and $\sigma_{\mu\nu}$ is the shear tensor. The following conditions are satisfied by these tensors :

$$u^\mu u_\mu = -1,\;\; u^\mu q_\mu = 0,\;\; \chi^\mu\chi_\mu = 1,\;\; \chi^\mu u_\mu = 0,\;\; l^\mu u_\mu = -1,\;\; l^\mu l_\mu = 0. \tag{8.13}$$

Assuming the observer to be comoving with the collapsing matter, we have

$$u^\mu = A^{-1}\delta_0^\mu,\;\; \chi^\mu = B^{-1}\delta_1^\mu,\;\; l^\mu = A^{-1}\delta_0^\mu + B^{-1}\delta_1^\mu. \tag{8.14}$$

It is further considered that the collapsing matter ball is charged, as a result of which, Maxwell's electromagnetic stress-energy tensor needs to be taken in consideration, in addition to the usual matter energy-momentum tensor. The electromagnetic stress-energy tensor is given by

$$E_{\mu\nu} = \frac{1}{4\pi}\left(-F^{\gamma}_{\mu}F_{\gamma\nu} + \frac{1}{4}F^{\delta\gamma}F_{\delta\gamma}g_{\mu\nu}\right) \tag{8.15}$$

where, $F_{\mu\nu} = \varphi_{\nu,\mu} - \varphi_{\mu,\nu}$ is the Maxwell's electromagnetic field tensor, with $\varphi_\mu$ being the electromagnetic four-potential, which, in the absence of magnetic field has the scalar potential $\varphi$ as its only non-zero component. The components representing the vector potential all vanish in the absence of magnetic field.

The effective energy-momentum tensor is given by

$$T^{eff}_{\mu\nu} = \frac{1}{f_R}\left[(1+f_T)\,T_{\mu\nu} - L_{m_{int}}g_{\mu\nu}f_T + \frac{(f-Rf_R)}{2}g_{\mu\nu} - D^{-}_{\mu\nu} + 8\pi E_{\mu\nu}\right] \tag{8.16}$$

where $D_{\mu\nu} = (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\,f_R$ represents the dark energy source terms, and the negative sign in the superscript denotes the interior spacetime. The Maxwell field equations are given by

$$F_{[\mu\nu;\gamma]} = 0, \tag{8.17}$$

and,

$$F^{\mu\nu}_{;\nu} = 4\pi J^{\mu}. \tag{8.18}$$

where $J^{\alpha} = ju^{\alpha}$ represents the four-current, with $j$ being the current density. The Maxwell field equations are

$$\varphi'' + \varphi'\left(-\frac{A'}{A} - \frac{B'}{B} + \frac{2C'}{C}\right) = 4\pi AB^2 j, \tag{8.19}$$

$$\dot{\varphi}' - \varphi'\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{2\dot{C}}{C}\right) = 0 \tag{8.20}$$

The conservation of charge requires

$$J^{\mu}_{;\mu} = 0, \tag{8.21}$$

which yields the total charge, $Q(r)$ inside the collapsing sphere as

$$Q(r) = 4\pi\int_0^r jBC^2 dr. \tag{8.22}$$

It can also be seen that

$$\frac{\varphi'^2}{A^2B^2}=\frac{Q^2}{C^4}. \tag{8.23}$$

Following the works of other authors [31, 52, 142], the mass-function, which describes the total energy contained inside a 3D timelike hypersurface acting as the boundary of the collapsing matter, is given by

$$m=\frac{C}{2}\left(1+\frac{\dot{C}^2}{A^2}-\frac{C'^2}{B^2}\right)+\frac{Q^2}{2C} \tag{8.24}$$

The expansion scalar $\Theta$ is given by

$$\Theta=\frac{1}{A}\left(\frac{\dot{B}}{B}+\frac{2\dot{C}}{C}\right), \tag{8.25}$$

and the shear scalar $\sigma$ is defined as

$$\sigma=-\frac{1}{3A}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right). \tag{8.26}$$

The structure scalars in terms of the metric coefficients, the function $m$ and the shear and expansion scalars, are given by

$$X_T=\frac{2\dot{C}\dot{B}}{CBA^2}-\frac{2C''}{CB^2}+\frac{2C'B'}{CB^3}+\frac{2m}{C^3}-\frac{Q^2}{C^4} \tag{8.27}$$

$$X_{TF}=-\frac{\dot{C}\dot{B}}{CBA^2}+\frac{C''}{CB^2}-\frac{C'B'}{CB^3}+\frac{2m}{C^3}-\frac{Q^2}{C^4} \tag{8.28}$$

$$Y_T=\frac{1}{B^2}\left(\frac{A''}{A}-\frac{A'B'}{AB}+\frac{2A'C'}{AC}\right)-\frac{1}{A^2}\left(\frac{\ddot{B}}{B}+\frac{2\ddot{C}}{C}\right)+\frac{\dot{A}\Theta}{A^2} \tag{8.29}$$

$$Y_{TF}=\frac{1}{B^2}\left(\frac{A''}{A}-\frac{A'B'}{AB}-\frac{A'C'}{AC}\right)-\frac{1}{A^2}\left(\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}\right)-\frac{3\dot{A}\sigma}{A^2} \tag{8.30}$$

The electric part of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ is given by $\mathcal{E}_{\alpha\beta}=C_{\alpha\gamma\beta\delta}u^\gamma u^\delta$, and the magnetic part is given by $H_{\alpha\beta}=\frac{1}{2}\eta_{\alpha\gamma\epsilon\rho}C^{\epsilon\rho}_{\beta\delta}u^\gamma u^\delta$. For the spherically symmetric case, all components of $H_{\alpha\beta}$ are zero and the magnetic part of the Weyl tensor vanishes.

The electric part of the Weyl tensor can also be expressed in terms of the Weyl scalar $\mathcal{E}$ as

$$\mathcal{E}_{\alpha\beta} = \mathcal{E}\left(\chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha\beta}\right) \tag{8.31}$$

It satisfies the properties :

$$\mathcal{E}^\alpha_\alpha = 0, \quad \mathcal{E}_{\alpha\beta} = \mathcal{E}_{(\alpha\beta)}, \quad \mathcal{E}_{\alpha\beta} u^\beta = 0. \tag{8.32}$$

The non-zero components of $\mathcal{E}_{\alpha\beta}$ [144], written in terms of the mass-function and the shear scalar, are found to be as follows :

$$\begin{aligned}\mathcal{E}_{11} &= -\frac{B^2}{3}\left[\frac{2m}{C^3} - \frac{Q^2}{C^4} + \frac{1}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C}\right) - \frac{1}{B^2}\left(\frac{A''}{A} - \frac{C''}{C}\right) + \frac{B'}{B^3}\left(\frac{A'}{A} - \frac{C'}{C}\right)\right. \\ &\left. + \frac{3\dot{A}\sigma}{A^2} + \frac{C'A'}{AB^2C} - \frac{\dot{C}\dot{B}}{A^2BC}\right]\end{aligned} \tag{8.33}$$

$$\begin{aligned}\mathcal{E}_{22} &= \frac{C^2}{6}\left[\frac{2m}{C^3} - \frac{Q^2}{C^4} + \frac{1}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C}\right) - \frac{1}{B^2}\left(\frac{A''}{A} - \frac{C''}{C}\right) + \frac{B'}{B^3}\left(\frac{A'}{A} - \frac{C'}{C}\right)\right. \\ &\left. + \frac{3\dot{A}\sigma}{A^2} + \frac{C'A'}{AB^2C} - \frac{\dot{C}\dot{B}}{A^2BC}\right]\end{aligned} \tag{8.34}$$

$$\begin{aligned}\mathcal{E}_{33} &= \frac{C^2\sin^2\theta}{6}\left[\frac{2m}{C^3} - \frac{Q^2}{C^4} + \frac{1}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C}\right) - \frac{1}{B^2}\left(\frac{A''}{A} - \frac{C''}{C}\right) + \frac{B'}{B^3}\left(\frac{A'}{A} - \frac{C'}{C}\right)\right. \\ &\left. + \frac{3\dot{A}\sigma}{A^2} + \frac{C'A'}{AB^2C} - \frac{\dot{C}\dot{B}}{A^2BC}\right]\end{aligned} \tag{8.35}$$

The Weyl Scalar $\mathcal{E}$ is given by

$$\begin{aligned}\mathcal{E} &= -\frac{1}{2}\left[\frac{2m}{C^3} - \frac{Q^2}{C^4} + \frac{1}{A^2}\left(\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C}\right) - \frac{1}{B^2}\left(\frac{A''}{A} - \frac{C''}{C}\right) + \frac{B'}{B^3}\left(\frac{A'}{A} - \frac{C'}{C}\right)\right. \\ &\left. + \frac{3\dot{A}\sigma}{A^2} + \frac{C'A'}{AB^2C} - \frac{\dot{C}\dot{B}}{A^2BC}\right]\end{aligned} \tag{8.36}$$

which, with the help of equations (8.28) and (8.30), can be expressed as

$$\mathcal{E} = \frac{1}{2}\left(Y_{TF} - X_{TF}\right) \tag{8.37}$$

On evaluating $Z = \sqrt{Z^{\alpha\beta}Z_{\alpha\beta}}$ , from equation (8.8), we find that

$$Z = \left[\left(\frac{2m}{C^3} - \frac{Q^2}{C^4}\right)^2 + 2\left(\frac{2m}{C^3} - \frac{Q^2}{C^4} - X_{TF}\right)^2\right]^{\frac{1}{2}} \tag{8.38}$$

## 8.5 The Field Equations for the Interior Spacetime

We consider our matter Lagrangian to be $-\rho$ , and using this in our $f(R,T)$ field equation, the effective energy-momentum tensor is given by

$$T^{eff}_{\mu\nu} = \frac{1}{f_R}\left[\left(1+f_T\right)T_{\mu\nu} + \rho g_{\mu\nu} f_T + \frac{(f - Rf_R)}{2} g_{\mu\nu} - D_{\mu\nu} + 8\pi E_{\mu\nu}\right] \tag{8.39}$$

where $D_{\mu\nu} = \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right) f_R$ represents the dark energy source terms. The field equations are found to be

$$G_{00} = \frac{A^2}{f_R}\left[\rho + \epsilon + \epsilon f_T - \frac{(f - Rf_R)}{2} - \frac{D_{00}}{A^2} + \frac{3Q^2}{C^4}\right] \tag{8.40}$$

$$G_{01} = \frac{AB}{f_R}\left[-\left(1+f_T\right)\left(q+\epsilon\right) - \frac{D_{01}}{AB}\right] \tag{8.41}$$

$$G_{11} = \frac{B^2}{f_R}\left[\left(1+f_T\right)\left(p_r + \epsilon + 4\eta\sigma\right) + \rho f_T + \frac{(f - Rf_R)}{2} - \frac{D_{11}}{B^2} - \frac{3Q^2}{C^4}\right] \tag{8.42}$$

$$G_{22} = \frac{C^2}{f_R}\left[\left(1+f_T\right)\left(p_\perp - 2\eta\sigma\right) + \rho f_T + \frac{(f - Rf_R)}{2} - \frac{D_{22}}{C^2} - \frac{Q^2}{C^4}\right] \tag{8.43}$$

Here the coefficient of shear viscosity, $\eta$, is positive. But the shear scalar is a negative quantity, as given by (8.26). As it can be seen from (8.42) and (8.43), the shear causes a decrease in the radial pressure, but contributes positively to the tangential pressure, which is natural, as shear viscosity arises out of different fluid layers moving against one another in the tangential direction. Further, for the situation where bulk viscosity might be present, the bulk viscosity will bring a negative contribution, opposing both the radial and tangential pressures. It is possible that for certain values, it might make the effective pressure negative, which would imply a repulsive gravity or dark energy effect. Hence bulk viscosity may possibly account for the accelerated expansion of the universe in this manner [229]. The dark source terms are given by

$$D_{00} = \dot{f}_R\left(\frac{2\dot{C}}{C}+\frac{\dot{B}}{B}\right) - \frac{A^2}{B^2}f_R'' - \frac{f_R'A^2}{B^2}\left(\frac{2C'}{C}-\frac{B'}{B}\right) \tag{8.44}$$

$$D_{01} = \frac{f_R'\dot{B}}{B} + \frac{\dot{f}_R A'}{A} - \dot{f}_R{}' \tag{8.45}$$

$$D_{11} = f_R'\left(\frac{A'}{A}+\frac{2C'}{C}\right) - \ddot{f}_R\frac{B^2}{A^2} - \frac{\dot{f}_R B^2}{A^2}\left(\frac{2\dot{C}}{C}-\frac{\dot{A}}{A}\right) \tag{8.46}$$

$$D_{22} = -\frac{\dot{f}_R C^2}{A^2}\left(\frac{\dot{C}}{C}+\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right) - \ddot{f}_R\frac{C^2}{A^2} + \frac{C^2}{B^2}f_R'' + \frac{f_R'C^2}{B^2}\left(\frac{C'}{C}-\frac{B'}{B}+\frac{A'}{A}\right) \tag{8.47}$$

Likewise in [58, 220, 215, 217, 216], the following quantities are now introduced :

$$\bar{\rho} = \rho + \epsilon, \quad \bar{q} = q + \epsilon, \quad \bar{p_r} = p_r + \epsilon, \quad \bar{\Pi} = \bar{p_r} - p_\perp. \tag{8.48}$$

The temporal variation and spatial variation of a quantity are represented by the following operators defined as

$$D_T \equiv \frac{1}{A}\frac{\partial}{\partial t} \tag{8.49}$$

and

$$D_C \equiv \frac{1}{C'}\frac{\partial}{\partial r} \tag{8.50}$$

The collapse velocity $U$ is given by $U = \frac{\dot{C}}{A}$ which must be negative for collapse to occur. We also define $H = \frac{C'}{B}$. Utilising all these definitions, we find the temporal variation of the function $m$ to be

$$\begin{aligned} D_T m &= -\frac{C^2}{2f_R}\left[U\left\{(1+f_T)(\bar{p_r}+4\eta\sigma)+\rho f_T+\frac{(f-Rf_R)}{2}-\frac{D_{11}}{B^2}-\frac{3Q^2}{C^4}\right\}\right. \\ &\left. +H\left\{\bar{q}(1+f_T)+\frac{D_{01}}{AB}\right\}\right]-\frac{UQ^2}{C^2} \end{aligned} \tag{8.51}$$

while the spatial variation is given by

$$D_C m = \frac{C^2}{2f_R}\left[\bar{\rho} + \epsilon f_T - \frac{(f - Rf_R)}{2} - \frac{D_{00}}{A^2} + \frac{3Q^2}{C^4} + \frac{U}{H}\left\{\bar{q}\left(1 + f_T\right) + \frac{D_{01}}{AB}\right\}\right] + D_C\left(\frac{Q^2}{2C}\right) \tag{8.52}$$

Hence, the function $m$ can be expressed as

$$m = \int_0^r \frac{C^2}{2f_R}\left[\bar{\rho} + \epsilon f_T - \frac{(f - Rf_R)}{2} - \frac{D_{00}}{A^2} + \frac{3Q^2}{C^4} + \frac{U}{H}\left\{\bar{q}\left(1 + f_T\right) + \frac{D_{01}}{AB}\right\}\right] C' dr + \frac{Q^2}{2C}. \tag{8.53}$$

As we can see, the mass-energy content of the collapsing matter, which, by virtue of the field equations, is described by $m$, is increased by the presence of electric charge.

## 8.6 The Relation of the Matter Variables to the Structure Scalars

The various physical properties of the collapsing matter, which involve the energy density, pressure anisotropy, expansion and shear, can be related to the structure scalars formed out of the Riemann tensor which is a purely geometrical quantity. Utilising the field equations and the Weyl scalar, the structure scalars can be expressed as

$$X_T = \frac{1}{f_R}\left[\bar{\rho} + \epsilon f_T - \frac{(f - Rf_R)}{2} - \frac{D_{00}}{A^2} + \frac{3Q^2}{C^4}\right] \tag{8.54}$$

$$X_{TF} = -\mathcal{E} - \frac{1}{2f_R}\left[\left(1 + f_T\right)\left(\bar{\Pi} + 6\eta\sigma\right) - \frac{D_{11}}{B^2} + \frac{D_{22}}{C^2} - \frac{2Q^2}{C^4}\right] \tag{8.55}$$

$$Y_T = \frac{1}{2f_R}\left[3\rho f_T + \bar{\rho} + \epsilon f_T + \left(1 + f_T\right)\left(\bar{p_r} + 2p_\perp\right) + f - Rf_R - \frac{D_{00}}{A^2} - \frac{D_{11}}{B^2} - \frac{2D_{22}}{C^2} - \frac{2Q^2}{C^4}\right] \tag{8.56}$$

$$Y_{TF} = \mathcal{E} - \frac{1}{2f_R}\left[\left(1 + f_T\right)\left(\bar{\Pi} + 6\eta\sigma\right) - \frac{D_{11}}{B^2} + \frac{D_{22}}{C^2} - \frac{2Q^2}{C^4}\right] \tag{8.57}$$

It can be seen that the presence of electric charge causes an increment in the structure scalars $X_T$ and $Y_{TF}$, and a decrement in $X_{TF}$ and $Y_T$.

Now the matter variables are combined in the following manner to obtain quantites which can be termed as "effective energy density", "effective radial pressure", "effective tangential pressure" and "effective pressure anistropy", and are denoted with the superscript $\mathscr{P}$ :

$$\rho^{\mathscr{P}} = \bar{\rho} + \epsilon f_T - \frac{D_{00}}{A^2} \tag{8.58}$$

$$P_r^{\mathscr{P}} = \bar{p_r} + 4\eta\sigma - \frac{D_{11}}{B^2} \tag{8.59}$$

$$P_\perp^{\mathscr{P}} = p_\perp - 2\eta\sigma - \frac{D_{22}}{C^2} \tag{8.60}$$

$$\Pi^{\mathscr{P}} = P_r^{\mathscr{P}} - P_\perp^{\mathscr{P}} = \bar{\Pi} + 6\eta\sigma - \frac{D_{11}}{B^2} + \frac{D_{22}}{C^2} \tag{8.61}$$

Hence we have the Weyl scalar expressed in terms of the physical matter variables in the following manner,

$$\mathcal{E} = \frac{1}{2f_R}\left[\rho^{\mathscr{P}} - \frac{(f - Rf_R)}{2} + \frac{5Q^2}{C^4} + \frac{D_{11}}{B^2} - \frac{D_{22}}{C^2} - (1 + f_T)\left(\bar{\Pi} + 6\eta\sigma\right)\right] - I \tag{8.62}$$

where,

$$I = \frac{3}{2C^3}\int_0^r \frac{C^2}{f_R}\left[\rho^{\mathscr{P}} - \frac{(f - Rf_R)}{2} + \frac{3Q^2}{C^4} + \frac{U}{H}\left\{\hat{q} + \frac{\psi_q}{AB}\right\}\right] C' dr \tag{8.63}$$

with $\hat{q} = \bar{q}\,(1 + f_T)$ and $\psi_q = D_{01}$. Now, in terms of the effective matter variables, our structure scalars become

$$X_T = \frac{1}{f_R}\left[\rho^{\mathscr{P}} - \frac{(f - Rf_R)}{2} + \frac{3Q^2}{C^4}\right] \tag{8.64}$$

$$X_{TF} = -\frac{1}{2f_R}\left[\rho^{\mathscr{P}} - \frac{(f - Rf_R)}{2} + \frac{3Q^2}{C^4}\right] + I \tag{8.65}$$

$$Y_T = \frac{1}{2f_R}\left[3\rho f_T + \rho^{\mathscr{P}} + (f - Rf_R) + \frac{D_{11}}{B^2} + \frac{2D_{22}}{C^2} + (1 + f_T)\left(3\bar{p_r} - 2\bar{\Pi}\right) - \frac{2Q^2}{C^4}\right] \tag{8.66}$$

$$Y_{TF} = \frac{1}{2f_R}\left[\rho^{\mathscr{P}} - \frac{(f - Rf_R)}{2} - \frac{2D_{11}}{B^2} + \frac{2D_{22}}{C^2} + \frac{7Q^2}{C^4} - 2\left(1 + f_T\right)\left(\bar{\Pi} + 6\eta\sigma\right)\right] - I \quad (8.67)$$

From the expressions of the structure scalars in terms of the metric coefficients, and the Raychaudhuri equation [59], we see that two of our structure scalars can be expressed in the following manner :

$$-Y_T = \frac{\Theta^2}{3} + 6\sigma^2 + u^\alpha \Theta_{;\alpha} - a^\alpha_{;\alpha} \quad (8.68)$$

and

$$Y_{TF} = a^2 + \chi^\alpha a_{;\alpha} - \frac{aC'}{BC} + 2\Theta\sigma - 3u^\alpha \sigma_{;\alpha} - 3\sigma^2 \quad (8.69)$$

where, $a^\alpha$ is the acceleration vector. The first of these expressions show how the expansion of the collapsing spherical star evolves during the collapse, which is governed by the structure scalar $Y_T$. The second expression describes the shear evolution of the star and is controlled by the structure scalar $Y_{TF}$.

Utilising the expressions for the shear scalar $\sigma$, the expansion scalar $\Theta$, $X_{TF}$ , and taking the radial derivative of the expression for $X_{TF}$ , we get

$$\left[X_{TF} + \frac{\rho_{eff}}{2f_R}\right]' = -\frac{3C'}{C}X_{TF} + \frac{(\Theta + 3\sigma)}{2f_R}\left(\hat{q}B + \frac{\psi_q}{A}\right) + \frac{9C'}{C^4}\left(m - \frac{Q^2}{2C}\right) \quad (8.70)$$

with

$$\rho_{eff} = \rho^{\mathscr{P}} - \frac{(f - Rf_R)}{2} + \frac{3Q^2}{C^4}. \quad (8.71)$$

Equation (8.70) can also be rewritten in terms of $X_{TF}$ and $Z$ as

$$\left[X_{TF} + \frac{\rho_{eff}}{2f_R}\right]' = -\frac{3C'}{C}X_{TF} + \frac{(\Theta + 3\sigma)}{2f_R}\left(\hat{q}B + \frac{\psi_q}{A}\right) + \frac{9C'}{2C}\left[Z^2 - 2\left(\frac{2m}{C^3} - \frac{Q^2}{C^4} - X_{TF}\right)^2\right]^{\frac{1}{2}} \quad (8.72)$$

This expression contains information as to the effect of $X_{TF}$ as well as $Z$ on the energy-density inhomogeneity of the collapsing model, and relates the tidal forces and the heat flux, expansion scalar and the shear scalar. Hence $X_T$ can also be expressed as

$$X_T = \frac{\rho_{eff}}{f_R} \tag{8.73}$$

As it can be seen,

- $X_{TF}$ and the mass-function $m$ together along with the charge $Q$ influence the energy density inhomogeneity when there is no dissipation, as can be seen from equation (8.70). The structure scalar $Z$ is connected to $X_{TF}$, by virtue of equation (8.38). This enables us to see how $Z$ affects the heat dissipation, as well as the energy density inhomogeneity, by a recasting of equation (8.70) into equation (8.72).

- The evolution of the expansion scalar is controlled by $Y_T$, where the effect of charge is already included in the expression for $Y_T$, given by equation (8.66).

- The evolution of the shear scalar is controlled by $Y_{TF}$, where the effect of charge is present in the expression for $Y_{TF}$, given by equation (8.67).

- The effective homogeneous energy density is influenced by $X_T$.

- The pressure anisotropy is also influenced by both $Y_T$ and $Y_{TF}$.

- The electric charge $Q$ appears in the expressions for all four of the structure scalars, causing an increase in $X_T$ and $Y_{TF}$, and a decrease in $X_{TF}$ and $Y_T$.

### 8.6.1 Case of Constant R and T conditions for dust ball

By $R = constant = \tilde{R}$ and $T = constant = \tilde{T}$, we have

$$X_T = constant = \tilde{X}_T \tag{8.74}$$

$$Y_T = constant = \tilde{Y}_T \tag{8.75}$$

where $\tilde{R}$ and $\tilde{T}$ denote the values of the Ricci tensor and the Trace of the enegy-momentum tensor, at constant $R$ and $T$ conditions.

For the case of dust, except the energy density, all the other physical parameters, i.e., radial and tangential pressures, shear, free-streaming radiation, heat flux and charge will vanish, and $\bar{\rho}$ becomes the same as $\rho$. In that case, we have the structure scalars related to the matter in the following forms :

$$\tilde{X}_T = \frac{1}{f_R}\left[\rho - \frac{(f - Rf_R)}{2} - \frac{D_{00}}{A^2}\right] \tag{8.76}$$

$$\tilde{Y}_T = \frac{1}{2f_R}\left[(1 + f_T)(2\rho + T) + 2\rho f_T + f - Rf_R\right] \tag{8.77}$$

$$\tilde{X}_{TF} = -\mathcal{E} \tag{8.78}$$

$$\tilde{Y}_{TF} = \mathcal{E} \tag{8.79}$$

It can be seen that the sum of $X_{TF}$ and $Y_{TF}$ vanishes in this special case.

### 8.6.2 Choosing a linear form of the $f(R,T)$ function

Choosing a linear form of the $f(R,T)$ function, where $f(R,T) = R + \lambda T$ gives rise to a power-law type of scale factor. In this case, $f_R = 1$, $f_T = \lambda$, and $D_{\mu\nu} = 0$ . Utilising this functional form, the relations between the matter variables and the structure scalars take the following form :

$$X_T = \bar{\rho} + \epsilon\lambda - \frac{\lambda T}{2} + \frac{3Q^2}{C^4} \tag{8.80}$$

$$X_{TF} = -\mathcal{E} - \frac{1}{2}\left[(1 + \lambda)\left(\bar{\Pi} + 6\eta\sigma\right) - \frac{2Q^2}{C^4}\right] \tag{8.81}$$

$$Y_T = \frac{1}{2}\left[3\rho\lambda + \bar{\rho} + \epsilon\lambda + (1 + \lambda)(\bar{p_r} + 2p_\perp) + \lambda T - \frac{2Q^2}{C^4}\right] \tag{8.82}$$

$$Y_{TF} = \mathcal{E} - \frac{1}{2}\left[(1 + \lambda)\left(\bar{\Pi} + 6\eta\sigma\right) - \frac{2Q^2}{C^4}\right] \tag{8.83}$$

The "effective" matter variables after being combined together, now become :

$$\rho^{\mathscr{P}} = \bar{\rho} + \epsilon\lambda \tag{8.84}$$

$$P_r^{\mathscr{P}} = \bar{p_r} + 4\eta\sigma \tag{8.85}$$

$$P_{\perp}^{\mathscr{P}} = p_{\perp} - 2\eta\sigma \tag{8.86}$$

$$\Pi^{\mathscr{P}} = P_r^{\mathscr{P}} - P_{\perp}^{\mathscr{P}} = \bar{\Pi} + 6\eta\sigma \tag{8.87}$$

In terms of the effective matter variables, the structure scalars can be expressed as :

$$X_T = \rho^{\mathscr{P}} - \frac{\lambda T}{2} + \frac{3Q^2}{C^4} \tag{8.88}$$

$$X_{TF} = -\mathcal{E} - \frac{1}{2}\left[(1+\lambda)\,\Pi^{\mathscr{P}} - \frac{2Q^2}{C^4}\right] \tag{8.89}$$

$$Y_T = \frac{1}{2}\left[3\rho\lambda + \rho^{\mathscr{P}} + (1+\lambda)\left(P_r^{\mathscr{P}} + 2P_{\perp}^{\mathscr{P}}\right) + \lambda T - \frac{2Q^2}{C^4}\right] \tag{8.90}$$

$$Y_{TF} = \mathcal{E} - \frac{1}{2}\left[(1+\lambda)\,\Pi^{\mathscr{P}} - \frac{2Q^2}{C^4}\right] \tag{8.91}$$

### 8.6.3 Complexity Factor

Now we proceed to discuss the effect of complexity on the progress of collapse of the matter configuration under our consideration. Utilising equations (8.27)-(8.30), and (8.36), the Weyl scalar can be expressed as :

$$\mathcal{E} = \frac{Y_{TF}}{2} + \frac{1}{4C^3}\int C^3\left(\frac{G_{00}}{A^2}\right)' dr + \frac{3}{4C^3}\int \frac{\dot{C}C^2 G_{01}}{A^2} dr \tag{8.92}$$

From equation (8.40), it can be seen that the quantity $\frac{G_{00}}{A^2}$ in the second term of (8.92) is proportional to the interior energy density $\rho$. Hence its radial derivative must be proportional to $\rho'$ which is the inhomgeneity of energy density. Hence the Weyl scalar also depends on the inhomogeneity of the energy density. Utilising equation (8.92), equation (8.57) can be rewritten as

$$\begin{aligned} Y_{TF} \;=\; & \frac{1}{2C^3}\int C^3\left(\frac{G_{00}}{A^2}\right)' dr + \frac{3}{2C^3}\int \frac{\dot{C}C^2 G_{01}}{A^2} dr - \frac{1}{f_R}\Big[(1+f_T)\left(\bar{\Pi} + 6\eta\sigma\right) \\ & -\frac{D_{11}}{B^2} + \frac{D_{22}}{C^2} - \frac{2Q^2}{C^4}\Big] \end{aligned} \tag{8.93}$$

As it can be seen, from equations (8.40) and (8.41), in case of GR, and in absence of all dissipative and electromagnetic effects, for isotropic pressure and homogeneous energy density, that is, for zero complexity, the structure scalar $Y_{TF}$ vanishes. Hence this particular structure scalar is a reasonable choice for describing the complexity of the collapsing system, and is known as the complexity factor. It is worth mentioning that a vanishing complexity can also arise from the condition $Y_{TF} = 0$, which gives

$$\frac{1}{2C^3}\int C^3\left(\frac{G_{00}}{A^2}\right)' dr + \frac{3}{2C^3}\int \frac{\dot{C}C^2G_{01}}{A^2}dr - \frac{1}{f_R}\left[(1+f_T)\left(\bar{\Pi}+6\eta\sigma\right)\right.$$
$$\left. -\frac{D_{11}}{B^2} + \frac{D_{22}}{C^2} - \frac{2Q^2}{C^4}\right] = 0. \tag{8.94}$$

## 8.7 Exterior Spacetime

We have considered the exterior spacetime to described by the generalized Vaidya metric of outgoing radiation, whose line element is given by :

$$ds_+^2 = -\left(1 - \frac{2M(v,Y)}{Y}\right)dv^2 - 2dvdY + Y^2 d\Omega^2, \tag{8.95}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Here, $v$ is the retarded null coordinate, and $Y$ is the radial coordinate for the exterior spacetime. The mass-energy content inside a radius $Y$ at a time $v$ is described by the function $M(v,Y)$.

The exterior energy-momentum tensor is assumed to be a composite matter which is a combination of type-I and type-II fluids [165], and is expressed as follows:

$$T^+_{\mu\nu} = \mu l_\mu l_\nu + (\rho + P)\left(l_\mu n_\nu + l_\nu n_\mu\right) + Pg^+_{\mu\nu}, \tag{8.96}$$

where the first term represents a type-I fluid of density $\mu$, representing null-like matter or radiation photons, and the remaining terms together represent a type-II fluid, with $\rho$ being its density and $P$ being its isotropic pressure, which represents a perfect fluid component which can represent timelike/massive particles. Types of matter are further discussed in [171]. Here,

$$l_\mu = \delta^0_\mu, \tag{8.97}$$

$$\text{and} \qquad n_\mu = \frac{1}{2}\left(1 - \frac{2M(v,Y)}{Y}\right)\delta^0_\mu - \delta^1_\mu. \tag{8.98}$$

The trace of the exterior energy-momentum tensor is

$$T^{+} = 6P + 2\rho. \tag{8.99}$$

The $f(R,T)$ field equations for the exterior is given by :

$$R_{\mu\nu} = \frac{1}{f_R}\left[(1+f_T)\,T^{m}_{\mu\nu} - L_{m_{ext}} g_{\mu\nu} f_T + \frac{1}{2} g_{\mu\nu} f - D_{\mu\nu}\right], \tag{8.100}$$

which gives us, using (8.95)

$$\begin{aligned}
R_{00} &= \frac{1}{Y^2}\left(\left(\frac{\partial^2 M}{\partial Y^2}\right)(2M - Y) - 2\frac{\partial M}{\partial v}\right) \\
&= \frac{1}{f_R}\left[(1+f_T)\left(\mu + \rho_{ext}\left(1 - \frac{2M}{Y}\right)\right) + \left(1 - \frac{2M}{Y}\right) L_{m_{ext}} f_T\right. \\
&\quad \left. -\frac{1}{2}\left(1 - \frac{2M}{Y}\right) f - D^{+}_{00}\right], \qquad (8.101) \\
R_{11} &= 0 = \frac{-D^{+}_{11}}{f_R}, \qquad (8.102) \\
R_{22} &= 2\frac{\partial M}{\partial Y} = \frac{1}{f_R}\left[(1+f_T)\,PY^2 - L_{m_{ext}} f_T Y^2 + \frac{Y^2}{2} f - D^{+}_{22}\right], \qquad (8.103) \\
R_{33} &= R_{22}\sin^2\theta, \qquad (8.104) \\
R_{01} &= -\frac{1}{Y}\frac{\partial^2 M}{\partial Y^2} = \frac{1}{f_R}\left[(1+f_T)(-\rho_{ext} - 2P) + L_{m_{ext}} f_T - \frac{f}{2} - D^{+}_{01}\right]. \qquad (8.105)
\end{aligned}$$

where, $L_{m_{ext}}$ is the matter Lagrangian for the exterior spacetime, and a comma indicates a partial derivative. Here, the exterior dark source terms are given by

$$\begin{aligned}
D^{+}_{00} &= \frac{1}{Y^3}\left[f_{R,_{YY}}\left(-4M^2 Y + 4MY^2 - Y^3\right)\right. \\
&\quad + f_{R,_{Y}}\left(-2M_{,Y} MY + M_{,Y} Y^2 - M_{,v} Y^2 - 6M^2 + 7MY - 2Y^2\right) \\
&\quad \left. + f_{R,_{vY}}\left(-4MY^2 + 2Y^3\right) - f_{R,_{vv}} Y^3 + f_{R,_{v}}\left(M_{,Y} Y^2 - 5MY + 2Y^2\right)\right], \qquad (8.106) \\
D^{+}_{11} &= -f_{R,_{YY}}, \qquad (8.107) \\
D^{+}_{22} &= Y\left[f_{R,_{YY}}(-2M + Y) + f_{R,_{Y}}(-2M_{,Y} + 1) - 2f_{R,_{Yv}} Y - f_{R,_{v}}\right], \qquad (8.108) \\
D^{+}_{01} &= \frac{f_{R,_{YY}}}{Y^2}\left(2MY - Y^2\right) + \frac{f_{R,_{Y}}}{Y^2}\left(M_{,Y} Y + 3M - 2Y\right) + f_{R,_{Yv}} + \frac{2}{Y} f_{R,_{v}}. \qquad (8.109)
\end{aligned}$$

Using (8.102) and (8.107), we have

$$f_{R_{,YY}} = 0, \tag{8.110}$$

which simplifies the dark source terms in the following manner :

$$\begin{aligned}
D^{+}_{00} &= \frac{1}{Y^3}\left[f_{R_{,Y}}\left(-2M_{,Y}MY + M_{,Y}Y^2 - M_{,v}Y^2 - 6M^2 + 7MY - 2Y^2\right)\right.\\
&\quad \left. + f_{R_{,vY}}\left(-4MY^2 + 2Y^3\right) - f_{R_{,vv}}Y^3 + f_{R_{,v}}\left(M_{,Y}Y^2 - 5MY + 2Y^2\right)\right], \quad (8.111)\\
D^{+}_{11} &= 0, \quad (8.112)\\
D^{+}_{22} &= Y\left[f_{R_{,Y}}\left(-2M_{,Y}+1\right) - 2f_{R_{,Yv}}Y - f_{R_{,v}}\right], \quad (8.113)\\
D^{+}_{01} &= \frac{f_{R_{,Y}}}{Y^2}\left(M_{,Y}Y + 3M - 2Y\right) + f_{R_{,Yv}} + \frac{2}{Y}f_{R_{,v}}. \quad (8.114)
\end{aligned}$$

## 8.8 The Interior Field Equations

The matter-Lagrangian for the interior is kept unspecified for this part of the analysis, and denoted by $L_{m_{int}}$. The field equations for the interior spacetime, re-expressed in terms of the components of the Ricci tensor, are found to be

$$R^{-}_{00} = \frac{A^2}{f_R}\left[\left(1+f_T\right)\left(\rho+\epsilon\right) + L_{m_{int}}f_T - \frac{f}{2} - \frac{D^{-}_{00}}{A^2} + \frac{3Q^2}{C^4}\right] \tag{8.115}$$

$$R^{-}_{01} = \frac{AB}{f_R}\left[-\left(1+f_T\right)\left(q+\epsilon\right) - \frac{D^{-}_{01}}{AB}\right] \tag{8.116}$$

$$R^{-}_{11} = \frac{B^2}{f_R}\left[\left(1+f_T\right)\left(p_r+\epsilon+4\eta\sigma\right) - L_{m_{int}}f_T + \frac{f}{2} - \frac{D^{-}_{11}}{B^2} - \frac{3Q^2}{C^4}\right] \tag{8.117}$$

$$R^{-}_{22} = \frac{C^2}{f_R}\left[\left(1+f_T\right)\left(p_{\perp}-2\eta\sigma\right) - L_{m_{int}}f_T + \frac{f}{2} - \frac{D^{-}_{22}}{C^2} - \frac{Q^2}{C^4}\right] \tag{8.118}$$

The dark source terms are given by

$$D^{-}_{00} = \dot{f}_R\left(\frac{2\dot{C}}{C} + \frac{\dot{B}}{B}\right) - \frac{A^2}{B^2}f''_R - \frac{f'_R A^2}{B^2}\left(\frac{2C'}{C} - \frac{B'}{B}\right) \tag{8.119}$$

$$D_{01}^{-} = \frac{f'_R \dot{B}}{B} + \frac{\dot{f}_R A'}{A} - \dot{f_R}' \tag{8.120}$$

$$D_{11}^{-} = f'_R \left( \frac{A'}{A} + \frac{2C'}{C} \right) - \ddot{f}_R \frac{B^2}{A^2} - \frac{\dot{f}_R B^2}{A^2} \left( \frac{2\dot{C}}{C} - \frac{\dot{A}}{A} \right) \tag{8.121}$$

$$D_{22}^{-} = -\frac{\dot{f}_R C^2}{A^2} \left( \frac{\dot{C}}{C} + \frac{\dot{B}}{B} - \frac{\dot{A}}{A} \right) - \ddot{f}_R \frac{C^2}{A^2} + \frac{C^2}{B^2} f''_R + \frac{f'_R C^2}{B^2} \left( \frac{C'}{C} - \frac{B'}{B} + \frac{A'}{A} \right) \tag{8.122}$$

The Trace of the interior energy-momentum tensor is given by

$$T^{-} = -\rho + p_r + 2p_{\perp}. \tag{8.123}$$

For the case where bulk viscosity is present, the bulk viscosity term will be present in the Trace. It will oppose and reduce the net effective pressure. The effective pressure may also become negative in that case, mimicking the dark energy phase of the evolution of the universe when it undergoes late-time accelerated expansion. So, the bulk viscosity term can be vital in explaining this accelerated expanding phase [229].

## 8.9 Junction Conditions

The junction conditions, or the continuity of quantities such as the spacetime metric, the trace part and tracefree part of the extrinsic curvature tensor, Ricci scalar, trace of the energy-momentum tensor, and their derivatives with respect to the corresponding interior and exterior coordinates, are essential to the study of gravitational collapse. In GR, only the spacetime metric and the extrinsic curvature tensor components are required to be continuous for a smooth matching across the boundary of the collapsing matter [172, 173]. The additional quantities are required to be matched across the boundary when one moves to $f(R,T)$ gravity from GR. The junction conditions for $f(R,T)$ gravity were first presented by Rosa [230]. The 3D timelike hypersurface separating the interior and the exterior spacetimes, and forming the boundary of the collapsing matter, is given by

$$ds^2_{\Sigma} = -d\tau^2 + \mathcal{R}(\tau)^2 \left( d\theta^2 + \sin^2\theta d\phi^2 \right), \tag{8.124}$$

where, $\tau$, $\theta$ and $\phi$ are the hypersurface coordinates, later denoted by $\xi^i$ collectively. The junction conditions for a smooth matching in $f(R,T)$ gravity given by Rosa [230] are provided as

$$[g_{\mu\nu}]^+_- = 0, \tag{8.125}$$

$$\left[\tilde{K}_{ij}\right]^+_- = 0, \tag{8.126}$$

$$[K]^+_- = 0, \tag{8.127}$$

$$[R]^+_- = 0, \tag{8.128}$$

$$[T]^+_- = 0, \tag{8.129}$$

$$[\partial_\mu T]^+_- = 0, \tag{8.130}$$

$$[\partial_\mu R]^+_- = 0. \tag{8.131}$$

Here, $K$ and $\tilde{K}_{ij}$ are the trace part and trace-free parts of the extrinsic curvature tensor $K_{ij}$, whose expression is given by

$$K_{ij} = -N_\sigma \left( \frac{\partial^2 \psi^\sigma}{\partial \xi^i \partial \xi^j} + \Gamma^\sigma_{\alpha\beta} \frac{\partial \psi^\alpha}{\partial \xi^i} \frac{\partial \psi^\beta}{\partial \xi^j} \right), \tag{8.132}$$

where $\Gamma^\sigma_{\alpha\beta}$ are the Christoffel symbols for the spacetime under consideration, $N_\sigma$ is the normal to the hypersurface, and $\psi^\sigma$ are the coordinates of the 4D-spacetime.

Using the first three junction conditions, (8.125), (8.126) and (8.127), and the field equations, (8.116) and (8.117), we obtain,

$$-\frac{C}{2}\left[\frac{1}{f_R}\left((1+f_T)(p_r+4\eta\sigma-q) - L_{m_{int}} f_T + \frac{f-Rf_R}{2} - \frac{D^-_{01}}{AB} - \frac{D^-_{11}}{B^2} - \frac{3Q^2}{C^4}\right)\right]$$
$$+\frac{Q^2}{2C^3}|_\Sigma = \frac{1}{Y} M_{,Y}|_\Sigma \tag{8.133}$$

which brings out the relation between the radial pressure and heat flux at the boundary, along with the shear viscosity, in presence of the terms arising due to modified gravity. In absence of charge and shear viscosity, for Vaidya metric in GR, this relation reduces to $p_r = q$ at the boundary.

From (8.128) and (8.129), it can be seen that for the same functional form of the $f(R,T)$ function in both the interior and the exterior regions, we must have

$$f(R,T)^-|_\Sigma = f(R,T)^+|_\Sigma, \tag{8.134}$$

which also leads to the continuity of $f_R$ and $f_T$ at the boundary. Utilising the continuity of $f(R,T)$, $f_R$ and $f_T$ across the boundary, and the junction conditions, we find that the matter-Lagrangians for the two spacetimes are related in the following manner :

$$4f_T\left(L_{m_{int}}-L_{m_{ext}}\right)|_\Sigma=\frac{D_{00}^-}{A^2}-\frac{D_{11}^-}{B^2}-\frac{2D_{22}^-}{C^2}-2D_{01}^++\frac{2D_{22}^+}{C^2}-\frac{8Q^2}{C^4}|_\Sigma \tag{8.135}$$

In absence of charge, and an $f(R,T)$ function which is linear in both $R$ and $T$, the Lagrangians will match at the boundary, a fact which has been discussed in detail in [198].

Using (8.130) and (8.131), it can be seen that the continuity of the quantities $\partial_\mu f$, $\partial_\mu f_R$ and $\partial_\mu f_T$ across the boundary are necessary, if the same form of the $f(R,T)$ function is chosen for both the interior and the exterior regions. This leads us to the following two conditions :

$$\begin{aligned}&-4\frac{\partial}{\partial t}\left(L_{m_{int}}f_T\right)+\frac{\partial}{\partial t}\left(\frac{D_{00}^-}{A^2}-\frac{D_{11}^-}{B^2}-\frac{2D_{22}^-}{C^2}\right)-8\frac{\partial}{\partial t}\frac{Q^2}{C^4}|_\Sigma\\&=-4\frac{\partial}{\partial v}\left(L_{m_{ext}}f_T\right)+2\frac{\partial}{\partial v}\left(D_{01}^+-\frac{D_{22}^+}{Y^2}\right)|_\Sigma,\end{aligned} \tag{8.136}$$

and,

$$\begin{aligned}&-4\frac{\partial}{\partial r}\left(L_{m_{int}}f_T\right)+\frac{\partial}{\partial r}\left(\frac{D_{00}^-}{A^2}-\frac{D_{11}^-}{B^2}-\frac{2D_{22}^-}{C^2}\right)-8\frac{\partial}{\partial r}\frac{Q^2}{C^4}|_\Sigma\\&=-4\frac{\partial}{\partial Y}\left(L_{m_{ext}}f_T\right)+2\frac{\partial}{\partial Y}\left(D_{01}^+-\frac{D_{22}^+}{Y^2}\right)|_\Sigma,\end{aligned} \tag{8.137}$$

where the effect of charge and modified gravity is clearly brought out, in form of the terms involving $Q$ and the extra curvature terms. Equations (8.133), (8.135),(8.136) and (8.137) need to be adhered to for a smooth matching at the boundary in order for the collapse to be physically viable. Choosing $f(R,T)=R+2\lambda T$ reproduces the conditions for the matching of the matter-Lagrangians and their derivatives across the boundary which have been presented in equations (47) and (52) of [198].

## 8.10 Energy Conditions in $f(R,T)$ gravity

Energy conditions are essentially the requirement that the matter under consideration shows a physically realistic behaviour. This rules out any exotic matter contribution.

A discussion on energy conditions can be found in [171]. In order to find the energy conditions, following the procedure adopted by Kolassis [231], and using the effective energy-momentum tensor $T^{eff}_{\mu\nu}$ instead of the matter energy momentum tensor $T_{\mu\nu}$, we solve the eigenvalue equation

$$\left|T^{eff}_{\mu\nu} - \mu g_{\mu\nu}\right| = 0 \tag{8.138}$$

Let the quantities $\frac{T^{eff}_{\mu\nu}}{A^2}$, $\frac{T^{eff}_{\mu\nu}}{B^2}$, $\frac{T^{eff}_{\mu\nu}}{AB}$, and $\frac{T^{eff}_{\mu\nu}}{C^2}$ be denoted by $k_1$, $k_2$, $k_3$ and $k_4$ respectively for notational simplicity, where

$$k_1 = \frac{1}{f_R}\left[\rho + \epsilon + \epsilon f_T - \frac{(f - Rf_R)}{2} - \frac{D_{00}}{A^2} + \frac{3\varphi'^2}{A^2B^2}\right] \tag{8.139}$$

$$k_2 = \frac{1}{f_R}\left[(1 + f_T)\,(p_r + \epsilon + 4\eta\sigma) + \rho f_T + \frac{(f - Rf_R)}{2} - \frac{D_{11}}{B^2} - \frac{3\varphi'^2}{A^2B^2}\right] \tag{8.140}$$

$$k_3 = \frac{1}{f_R}\left[-(1 + f_T)\,(q + \epsilon) - \frac{D_{01}}{AB}\right] \tag{8.141}$$

$$k_4 = \frac{1}{f_R}\left[(1 + f_T)\,(p_\perp - 2\eta\sigma) + \rho f_T + \frac{(f - Rf_R)}{2} - \frac{D_{22}}{C^2} - \frac{\varphi'^2}{A^2B^2}\right] \tag{8.142}$$

The eigenvalues $\mu_0$, $\mu_1$, $\mu_2$ and $\mu_3$ are obtained from (8.138) as follows :

$$\mu_0 = \frac{k_2 - k_1 - \Delta}{2} \tag{8.143}$$

$$\mu_1 = \frac{k_2 - k_1 + \Delta}{2} \tag{8.144}$$

$$\mu_2 = \mu_3 = k_4 \tag{8.145}$$

where,

$$\Delta = \sqrt{(k_1 + k_2 + 2k_3)\,(k_1 + k_2 - 2k_3)} \tag{8.146}$$

From the condition that $\Delta^2$ has to be positive definite for a real value of $\Delta$, we have the condition,

$$|k_1 + k_2| - 2|k_3| \geq 0 \tag{8.147}$$

which is actually the null energy condition (NEC), stating that null vectors (which move along the path of photon) cannot carry negative energy density. The weak energy condition (WEC) requires that

$$-\mu_0 \geq 0; \tag{8.148}$$

$$-\mu_0 + \mu_i \geq 0 \tag{8.149}$$

where, i =1,2,3. This gives us the following conditions :

$$k_1 - k_2 + \Delta \geq 0 \tag{8.150}$$

and,

$$2k_4 - (k_2 - k_1 - \Delta) \geq 0 \tag{8.151}$$

These conditions require that the matter density is always non-negative for timelike observers. The dominant energy condition (DEC) requires that

$$\mu_0 \leq \mu_i \leq -\mu_0 \tag{8.152}$$

which gives us

$$k_1 - k_2 \geq 0 \tag{8.153}$$

and,

$$k_1 - k_2 + \Delta - 2k_4 \geq 0 \tag{8.154}$$

These conditions require that the flow of energy and momentum cannot exceed the velocity of light. The strong energy condition (SEC) requires that

$$-\mu_0 + \sum_i \mu_i \geq 0 \tag{8.155}$$

$$-\mu_0 + \mu_i \geq 0 \tag{8.156}$$

which gives us

$$\Delta + 2k_4 \geq 0 \tag{8.157}$$

The SEC requires that gravity is always attractive, and the sum of energy and effective pressures in every direction is non-negative. These equations can be written as follows :

NEC :

$$|\,(1+f_T)\,(\rho + p_r + 4\eta\sigma + 2\epsilon) - \frac{D_{00}}{A^2} - \frac{D_{11}}{B^2}| - 2|\,(1+f_T)\,(q+\epsilon) + \frac{D_{01}}{AB}| \geq 0 \tag{8.158}$$

WEC :

$$\frac{1}{f_R}\left[\rho\,(1-f_T) - (1+f_T)\,(p_r + 4\eta\sigma) - (f - Rf_R) - \frac{D_{00}}{A^2} + \frac{D_{11}}{B^2} + \frac{6Q^2}{C^4}\right] + \Delta \geq 0 \tag{8.159}$$

$$\frac{1}{f_R}\left[(1+f_T)\,(\rho - p_r + 2p_\perp - 8\eta\sigma) - \frac{D_{00}}{A^2} + \frac{D_{11}}{B^2} - \frac{2D_{22}}{C^2} + \frac{4Q^2}{C^4}\right] + \Delta \geq 0 \tag{8.160}$$

DEC :

$$\frac{1}{f_R}\left[\rho\left(1-f_T\right)-\left(1+f_T\right)\left(p_r+4\eta\sigma\right)-\left(f-Rf_R\right)-\frac{D_{00}}{A^2}+\frac{D_{11}}{B^2}+\frac{6Q^2}{C^4}\right]\geq 0 \tag{8.161}$$

$$\frac{1}{f_R}\left[\rho-3\rho f_T-\left(1+f_T\right)\left(p_r+2p_\perp\right)-2\left(f-Rf_R\right)-\frac{D_{00}}{A^2}+\frac{D_{11}}{B^2}+\frac{2D_{22}}{C^2}+\frac{8Q^2}{C^4}\right]+\Delta\geq 0 \tag{8.162}$$

SEC :

$$\frac{2}{f_R}\left[\left(1+f_T\right)\left(p_\perp-2\eta\sigma\right)+\rho f_T+\frac{\left(f-Rf_R\right)}{2}-\frac{D_{22}}{C^2}-\frac{\varphi'^2}{A^2B^2}\right]+\Delta\geq 0 \tag{8.163}$$

For the inclusion of bulk viscosity, the radial and tangential pressure components will get reduced by the effect of the bulk viscosity term. The possibility of an overall negative pressure owing to the presence of a bulk viscosity term, may lead to situations where the entire left hand side of the inequality given by the SEC becomes negative. In that case, the SEC is violated, and hence, gravity will become repulsive. This violation of SEC in case of $f(R,T)$ gravity may provide a possible explanation of the currently observed accelerated expanding phase of the universe.

Choosing a linear form of the $f(R,T)$ function, where $f(R,T) = R+\lambda T$, equation (8.158), which represents the NEC, can be plotted against the energy density $\rho$ and the radial pressure $p_r$ for various values of $\eta$, $q$, $\sigma$ and $\epsilon$. In terms of the structure scalars, the terms $k_1$, $k_2$, and $k_4$ can be expressed as follows :

$$k_1 = X_T \tag{8.164}$$

$$k_2 = \frac{1}{3}\left(2Y_T-2X_{TF}-2Y_{TF}-X_T\right) \tag{8.165}$$

$$k_4 = \frac{1}{3}\left(X_{TF}+Y_{TF}+2Y_T-X_T\right) \tag{8.166}$$

The term $k_3$ cannot be expressed in terms of any structure scalar.

Expressed in terms of the structure scalars, the energy conditions now take the following form :

WEC :

$$\frac{2}{3}\left(2X_T-Y_T+X_{TF}+Y_{TF}\right)+\Delta\geq 0 \tag{8.167}$$

$$\frac{2}{3}\left(X_T+Y_T+2X_{TF}+2Y_{TF}\right)+\Delta\geq 0 \tag{8.168}$$

DEC :

$$2X_T - Y_T + X_{TF} + Y_{TF} \geq 0 \tag{8.169}$$

$$2\left(X_T - Y_T\right) + \Delta \geq 0 \tag{8.170}$$

SEC :

$$\frac{2}{3}\left(2Y_T - X_T + X_{TF} + Y_{TF}\right) + \Delta \geq 0 \tag{8.171}$$

## 8.11 Results and Conclusions

In this paper, we have investigated the role of the structure scalars on the various matter variables for the collapse of a charged spherically symmetric shearing dissipative fluid in $f(R,T)$ gravity. The most general spherically symmetric spacetime has been considered for the interior of the collapsing matter. The structure scalars have been derived in terms of the spacetime geometry, and then, with the help of the $f(R,T)$ field equations, these have been related to the physical parameters of the matter, such as, energy density inhomogeneity, pressure anisotropy, shear viscosity, and charge. Initially, a general treatment has been provided without specifying the functional form of the $f(R,T)$ function, and later, the relation between the structure scalars and the matter variables have been provided for a linear form of the $f(R,T)$ function. We have also provided these relations in the special case of constant $R$ and $T$, for a relativistic dust ball. Choosing the exterior spacetime as the generalized Vaidya metric of outgoing radiation, the $f(R,T)$ junction conditions have been used to examine the relation between the radial pressure, heat flux, shear viscosity at the fluid boundary in presence of charge and modified gravity effects. Further, the relation between the matter Lagrangians of the exterior and the interior regions, along with their derivatives have also been found, in presence of charge and modified gravity effects. The energy conditions have been presented and the possibility of violation of the SEC in the context of $f(R,T)$ gravity has been discussed.

The following can be seen from the relation between the structure scalars and the matter variables given in equations (8.54), (8.55), (8.56) and (8.57) :

- In absence of dissipation, the energy density inhomogeneity is influenced by $X_{TF}$ and the mass-function $m$ together. As it can be seen from equation (8.70) in the presence of electromagnetic field, the charge plays a role in influencing the energy density inhomogeneity and $X_{TF}$.

- The heat dissipation and the energy density inhomogeneity is also controlled by $Z$ in addition to $X_{TF}$, as is evident from equation (8.72).
- The evolution of the expansion scalar is controlled by $Y_T$, as can be seen from equation (8.68). The contribution of the charge is manifested in the structure scalar $Y_T$ as is evident from equation (8.66).
- The evolution of the shear scalar is controlled by $Y_{TF}$, as can be seen from equation (8.69). The contribution of the charge is present in the structure scalar $Y_{TF}$, as can be seen from equation (8.67).
- The effective homogeneous energy density is influenced by $X_T$, as can be seen from equation (8.73).
- The pressure anisotropy is also influenced by both $Y_T$ and $Y_{TF}$.
- The electric charge $Q$ appears in the expressions for all four of the structure scalars, causing an increment in $X_T$ and $Y_{TF}$, and a decrement in $X_{TF}$ and $Y_T$.
- The presence of electric charge also causes an increase in the mass-energy content of the collapsing matter, as is evident from equation (8.53).
- The structure scalar $Y_{TF}$ can also be considered as the complexity factor, as it vanishes in case of isotropic pressure and homogeneous energy density in absence of modified gravity. The condition for vanishing complexity has also been obtained in equation (8.94), for which $Y_{TF}$ becomes zero.

To the best of our knowledge, although the influence of electromagnetic field on the structure scalars had previously been discussed in the context of GR [204], $f(R)$ gravity [216], $f(R,T,Q)$ gravity [201], and similar other theories, but it was not done for $f(R,T)$ gravity. Unlike the case in GR [204], we have not absorbed the contribution from the charge into the effective matter variables (superscripted by $\mathscr{P}$), but kept it as a separate entity, in order to clearly identify the increment and decrement of the structure scalars due to the presence of the electromagnetic factor, which, as it can also be seen from equations (8.64)-(8.67), differ by their integer coefficients. This is a significant outcome of our study.

For the choice of an $f(R,T)$ function of the form $f(R) + \lambda T$, where $f(R) = R + \alpha R^2$, which is the Starobinsky model [232, 233, 234], it can clearly be seen from the analysis of the junction conditions, that the higher order curvature terms in this case will play a significant role in constraining the matching conditions for the interior and exterior matter-Lagrangians for the collapse to be feasible, which in case of a linear functional model like $R + \lambda T$, and absence of charge and shear, will just result in the continuity of the matter Lagrangian across the boundary. Also for the case of a constant scalar curvature $R_0$, the higher order curvature terms would vanish. Black hole solutions in the presence of the cosmological constant can be obtained for constant scalar curvature and constant trace choices of the $f(R,T)$ function, in a way similar to the $f(R)$ theory case described in [235]. Further, like the $f(R)$ junction conditions discussed in [236], we have discussed the consequences of the $f(R,T)$ junction conditions which have the additional constraints of matching the Ricci scalar and its normal derivative, and the Trace of the energy-momentum tensor and its normal derivative across the boundary of the collapsing sphere for a smooth matching. These extra conditions have led us to the constraints on the interior and exterior matter-Lagrangians given by equations (8.135), (8.136) and (8.137). Further the choice of our exterior spacetime as the generalized Vaidya spacetime for outgoing radiation has led to the further constraint on the choice of our $f(R,T)$ function which is given by equation (8.110).

We have plans to extend this piece of investigation to other configurations of collapsing matter in modified gravity for different exterior spacetimes, and also for different geometry.

"

# Part III

# Summary and Outlook

We now discuss the overall takeaways from the work presented in this thesis in its entirety, and possible future directions.

This thesis deals with the phenomenon of gravitational collapse within the framework of modified gravity, specifically, $f(R,T)$ gravity. Different facets of this astrophysical phenomenon, viz., dynamical conditions which determine the stability/instability of the collapse, the causal transport of heat and its effect on the collapse, the junction conditions across the boundary of a collapsing matter ball, the formation of the final singularity and the time taken for its formation, the possibility of the formation of an apparent horizon, and whether it forms before or after the final singularity, thus deciding whether the final singularity is a black hole or a naked singularity, respectively, and the effect of "structure scalars" [202] on the physical properties of the collapsing matter, such as the enrgy density inhomogeneity, pressure anisotropy, heat flux, shear, expansion, and the effect of charge on these scalars, have been investigated for a variety of dissipative matter configurations, some involving shear viscosity and free streaming radiation, and the others in absence of these effects. We have also identified the structure scalar which provides us the measure of complexity of the collapsing system under consideration.

We have studied the progression of collapse for the most general spherically symmetric relativistic fluid involving pressure anisotropy, dissipation in the form of radial heat flow, free-streaming radiation and shear viscosity, and found the bounds of the adiabatic index within which the stability of the matter configuration is maintained during the process, both in the Newtonian and the post-Newtonian regimes, in the framework of $f(R,T)$ gravity, where we have chosen an $f(R,T)$ function which combines a linear term involving the Trace $T$ with the Starobinsky $f(R)$ function which is quadratic in the Ricci scalar. The perturbation scheme has been utilised in this portion of the analysis. Making use of the Israel-Stewart theory of causal heat transport, we have obtained the transport equations, and coupling them with the dynamical equation, we have confirmed the validity of the weak equivalence principle for the collapse mechanism, and identified the point of transition between expansion and collapse of the system. Dynamical analysis, and causal transport had previously been investigated in GR, $f(R)$ and $f(R,T)$ gravity, either for simpler matter configurations, or for simpler (linear) modified gravity function in $f(R,T)$, but not with a combination of the most generalised matter and a quadratic $f(R,T)$ function at the same time. We have not considered the presence of charge in our analysis however, and it leaves room for future works regarding this factor.

We have also considered the spherically symmetric collapse of an isotropic fluid undergoing dissipation in the form of heat flux in the radial direction, for a linear

$f(R,T)$ function. Because of the presence of dissipative effects, we have taken the exterior spacetime to be that of a generalised Vaidya type, and filled with a combination of Type-I and Type-II fluids. Utilising the junction conditions for $f(R,T)$ gravity, we have verified an assumption made by Rosa [230] that the interior and the exterior matter Lagrangians must match at the boundary of the collapsing sphere. The time of formation of the final singularity has been determined, followed by the time of formation of apparent horizon, and conclusions have been drawn about the constraints on parameters (arising out of the solutions for these two time values) for the final singularity to be a black hole. A lot of previous research in modified theory has made the same analysis using the Darmois-Israel junction conditions which are valid for GR but not for $f(R,T)$ gravity, unlike our case, where we have utilised the correct junction conditions. Further, solutions of the metric coefficients had been found for charged and uncharged perfect fluids in GR, and modified theories, but not for dissipative matter in previously existing literature. Owing to the fact that we have considered heat flux in our analysis, unlike the previous examples, the $f(R,T)$ field equation for the $G_{01}$ component of the Einstein tensor cannot be solved unless we know the exact functional form of the heat flux $q(r,t)$. The differential equation arising from the pressure isotropy condition is a highly complicated non-linear 2nd-order differential equation. We have had to resort to assumptions of separable forms of the metric coefficients in the radial and temporal coordinates, following [27, 28, 35], in order to obtain solutions. We have not considered shear in our analysis, which would have resulted in a more complicated form of the differential equations, but attempts might be made in the future to carry out this analysis in presence of shearing effects and examine how the constraints for the final singularity get modified. Also, a quadratic term in the Ricci scalar in the $f(R,T)$ function would result in the higher order curvature terms being present in the differential equation arising from the pressure isotropy condition, all of which would involve knowing the functional dependence of the terms on the radial and temporal coordinates, in order that a solution be attainable.

Lastly, we have investigated how the structure scalars affect the different physical properties of a collapsing matter, which we have considered to be the most generalised in composition, and spherically symmetric in geometry. Without specifying the form of the $f(R,T)$ function, a general treatment was presented. The influence of the various structure scalars, in combination, or exclusively, on the energy density inhomogeneity, pressure anisotropy, heat dissipation, homogeneous energy density, and the evolution of expansion and shear of the collapsing matter have been expressed. Unlike in previous literature however, the influence of charge is clearly brought out by keeping the charge separate from the terms describing other physical properties, and not absorbing it into the effective representations of the energy density and pressure. It is also highlighted how the mass-energy content of the collapsing sphere increases in presence of the charge. The charge also affects the structure scalars $X_T$, $X_{TF}$, $Y_T$ and $Y_{TF}$. A special case was also shown regarding a linear $f(R,T)$ functional choice, and for a relativistic dust ball. Finally, we have showed and justified that the structure scalar $Y_{TF}$ provides us the measure of complexity of the collapsing matter because it vanishes in the case of the simplest collapsing configuration, one that of isotropic pressure and homogeneous energy density. To identify whether the self-gravitating matter under consideration exhibits physically realistic behaviour or not, we have also examined the energy conditions for this collapsing system and have analysed the situations under which the Strong Energy Condition will be satisfied by the system.

In none of the analyses mentioned, have we investigated the constraints on the $f(R,T)$ coupling parameter, which remains an area for possible future studies. The aspects discussed also remain to be investigated in a geometry which is not spherically symmetric, for example, cylindrical or plane-symmetric. All these problems may be carried over to a far more general theory of modified gravity, for example, the $f(R,T,R_{\mu\nu}T^{\mu\nu})$ theory, with higher degrees of freedom, in the future. Energy conditions and their modifications from GR, when we shift to modified gravity, is also a direction which remains to be explored, although some preliminary work has already been done in this area [203]. Hence, this thesis serves as just the tip of a vast iceberg, whose secrets remain yet to be unravelled.

# Bibliography


[1] A. Einstein, (1915) On the General Theory of Relativity. Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.), 1915, 778-786.

[2] A. De Felice, S. Tsujikawa, $f(R)$ Theories. Living Rev. Relativ. **13**, 3 (2010). https://doi.org/10.12942/lrr-2010-3.

[3] Supernova Search Team Collaboration, A. G. Riess et al., Astron. J. **116**, 1009 (1998).

[4] S. Perlmutter, et al., Astrophys. J. **517**, 565 (1999).

[5] A. Friedmann, Zeitschrift fur Physik, Über die Krümmung des Raumes, **10**, 377 (1922). doi:10.1007/BF01332580.

[6] H. A. Buchdahl, Mon. Not. R. Astron. Soc. **150**, 1 (1970). https://doi.org/10.1093/mnras/150.1.1 .

[7] P. Sotiriou, V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010).

[8] S. Capozziello, M. De Laurentis, Phys. Rept. **509**, 167 (2011).

[9] T. Harko, F. S. N. Lobo, S. Nojiri, S. D. Odinstov, Phys. Rev. D **84**, 024020 (2011).

[10] X. Yang, D. Chen, Mon. Not. R. Astron. Soc. **394**, 1449 (2009).

[11] T. Harko, F. S. N. Lobo, Eur. Phys. J. C **70**, 373 (2010).

[12] Z. Haghani, T. Harko, F. S. N. Lobo, H. R. Sepangi, S. Shahidi, Phys. Rev. D **88**, 044023 (2013).

[13] S. D. Odintsov, D. Sáez-Gómez, Phys. Lett. B **725**, 437 (2013).

[14] C. H. Brans, R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[15] D. Lovelock, J. Math. Phys. **12**, 498 (1971).

[16] J. R. Oppenheimer, H. Snyder, Phys. Rev. **56**, 455 (1939).

[17] B. Datt, Zeitschrift für Physik **108**, 314 (1938).

[18] E. N. Glass, Phys. Lett. A **86**, 351 (1981).

[19] N. O. Santos, Mon. Not. R. Astron. Soc. **216**, 403 (1985).

[20] A. K. G. de Oliveira, J. A. de F. Pacheco, N. O. Santos, Mon. Not. R. astr. Soc. **220**, 405 (1986).

[21] A. K. G. de Oliveira and N. O. Santos, Astrophys. J. **312** 640 (1987).

[22] A. K. G. de Oliveira, C. A. Kolassis and N. O. Santos, Mon. Not. R. astr. Soc. **231**, 1011 (1988).

[23] L. Herrera, G. Le Denmat, N.O. Santos, Mon. Not. R. Astron. Soc. **237**, 257 (1989).

[24] R. Chan, S. Kichenassamy, G. Le Denmat, N.O. Santos, Mon. Not. R. Astron. Soc. **239**, 91 (1989).

[25] R. Chan, L. Herrera, N.O. Santos, Mon. Not. R. Astron. Soc. **265**, 533 (1993).

[26] R. Chan, L. Herrera, N.O. Santos, Mon. Not. R. Astron. Soc. **267**, 637 (1994).

[27] R. Chan, Mon. Not. R. Astron. Soc. **288**, 589 (1997).

[28] R. Chan, Mon. Not. R. Astron. Soc. **316**, 588 (2000).

[29] L. Herrera, N. O. Santos, Phys. Rev. D, Vol. **70**, 084004 (2004).

[30] R. Goswami, P. S. Joshi, Phys. Rev. D **69**, 027502 (2004).

[31] A. Di Prisco, L. Herrera, G. Le Denmat, M. A. H. MacCallum, N. O. Santos, Phys. Rev. D **76**, 064017 (2007).

[32] B. C. Tewari, Astrophys. Space Sci. **342**, 73 (2012).

[33] B. C. Tewari, Gen. Relativ. Gravit. **45**, 1547 (2013).

[34] B. C. Tewari, K. Charan, Astrophys. Space Sci. **351**, 613 (2014).

[35] S. Guha, R. Banerji, Int. J. Theor. Phys. **53**, 2332 (2014).

[36] M. Sharif, H. R. Kausar, Astrophys. Space Sci. **331**, 281 (2011).

[37] M. Sharif, H. R. Kausar, JCAP **07**, 022 (2011).

[38] M. Sharif, Z. Yousaf, Phys. Rev. D **88**, 024020 (2013).

[39] M. Sharif, Z. Yousaf, Mon. Not. R. Astron. Soc. **432**, 264 (2013).

[40] M. Sharif, Z. Yousaf, Mon. Not. R. Astron. Soc. **434**, 2529 (2013).

[41] M. Sharif and Z. Yousaf, Mon. Not. Roy. Astron. Soc. **440**, 3479 (2014).

[42] I. Noureen, A. A. Bhatti, M. Zubair, JCAP **02**, 033 (2015).

[43] S. Chakrabarti, N. Banerjee, Gen. Relativ. Gravit. **48**, 57 (2016).

[44] S. Chowdhury, K. Pal, K. Pal, T. Sarkar, Eur. Phys. J. C **80**, 902 (2020).

[45] M. Sharif, Z. Yousaf, Astrophys. Space Sci. **354**, 471 (2014).

[46] I. Noureen, M. Zubair, Astrophys. Space Sci. **356**, 103 (2015).

[47] I. Noureen, M. Zubair, Eur. Phys. J. C **75**, 62 (2015).

[48] I. Noureen, M. Zubair, A. A. Bhatti, G. Abbas, Eur. Phys. J. C **75**, 323 (2015).

[49] Z. Yousaf, M. Z. Bhatti, Eur. Phys. J. C **76**, 267 (2016).

[50] Z. Yousaf, K. Bamba, M. Z. Bhatti, Phys. Rev. D **93**, 124048 (2016).

[51] M. J. Amir, S. Sattar, Int. J. Theor. Phys. **55**, 1988 (2016).

[52] M. Sharif, A. Waseem, Gen. Relativ. Gravit. **50**, 78 (2018).

[53] G. Abbas, R. Ahmed, Mod. Phys. Lett. A **34**, 1950153 (2019).

[54] S. Guha and U. Ghosh, **Eur. Phys. J. Plus (2021) 136:460**; DOI : **10.1140/epjp/s13360-021-01446-4**.

[55] J. L. Rosa, Phys. Rev. D **103**, 104069 (2021).

[56] L. Bel, Ann. Henri Poincaré **17**, 37 (1961).

[57] A. Garcia-Parrado Gomez-Lobo, Class. Quantum Grav. **25**, 015006 (2008).

[58] L. Herrera, J. Ospino, A. Di Prisco, E. Fuenmayor, O. Troconis, Phys. Rev. D **79**, 064025 (2009).

[59] A. Raychaudhuri, Relativistic cosmology, I, Phys. Rev. **98**, 1123 (1955). Reprinted as a "Golden Oldie" in Gen. Relativ. Gravit. **32**, 749 (2000).

[60] S. Chandrasekhar, Mon. Not. R. Astr. Soc. **91**, 456 (1931); ibid. **95**, 207 (1935); Astrophys. J. **74**, 81 (1931); Observatory **57**, 373 (1934).

[61] C. Hansen, S. Kawaler, *Stellar Interiors: Physical Principles, Structure and Evolution* (Springer Verlag, 1994).

[62] R. Kippenhahn, A. Weigert, *Stellar Structure and Evolution* (Springer Verlag, 1990).

[63] A. K. G. De Oliveira, N. O. Santos, C. A. Kolassis, Mon. Not. R. Astron. Soc. **216**, 1001 (1985).

[64] L. Herrera, N. O. Santos, G. Le Denmat, Gen. Relativ. Gravit. **44**, 1143 (2012).

[65] M. Govender, K. S. Govinder, S. D. Maharaj, Int. Jour. Mod. Phys. D **12**, 667 (2003).

[66] S. D. Maharaj, G. Govender, M. Govender, Gen. Relativ. Gravit. **44**, 1089 (2012).

[67] M. Govender, R. Maartens, S. D. Maharaj, Mon. Not. R. Astron. Soc. **310**, 557 (1999).

[68] S. D. Maharaj, G. Govender, M. Govender, Pramana - J. Phys. **77**, 469 (2011).

[69] S. Nojiri, S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007).

[70] S. Nojiri, S. D. Odintsov, Phys. Rept. **505**, 59 (2011).

[71] T. Clifton, P.G. Ferreira, A. Padilla, C. Skordis, Phys. Rept. **513**, 1 (2012).

[72] O. Bertolami, M. C. Sequeira, Phys. Rev. D **79**, 104010 (2009).

[73] T. Harko, F. S. N. Lobo, G. Otalora, E. N. Saridakis, Phys. Rev. D **89**, 124036 (2014).

[74] T. Harko, F. S. N. Lobo, G. Otalorac, E.N. Saridakis, JCAP **12**, 021 (2014).

[75] M. Sharif, M. Zubair, JHEP **12**, 079 (2013).

[76] S. Nojiri, S. Odintsov, Phys. Rev. D **74**, 086005 (2006).

[77] S. Nojiri, S. Odintsov, Gen. Relat. Gravit. **36**, 1765 (2004).

[78] S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002).

[79] S. M. Carroll et al., Phys. Rev. D **71**, 063513 (2005).

[80] S. Carloni, P. K. S. Dunsby, S. Capozziello, A. Troisi, Class. Quant. Gravity **22**, 4839 (2005).

[81] J. A. R. Cembranos, Phys. Rev. D **73**, 064029 (2006).

[82] T. Clifton, J.D. Barrow, Phys. Rev. D **72**, 103005 (2005).

[83] K. Bamba, C.Q. Geng, C.C. Lee, JCAP 08, 021 (2010).

[84] A. de la Cruz-Dombriz, A. Dobado, Phys. Rev. D **74**, 087501 (2006).

[85] P.K.S. Dunsby, E. Elizalde, R. Goswami, S. Odintsov, D. Sáez-Gómez, Phys. Rev. D **82**, 023519 (2010).

[86] S. Nojiri, S.D. Odintsov, Phys. Rev. D 68, 123512 (2003)

[87] S. G. Ghosh, S.D. Maharaj, Phys. Rev. D **85**, 124064 (2012).

[88] J. A. R. Cembranos, A. de la Cruz-Dombriz, B. M. Nunez, JCAP **04**, 021 (2012).

[89] R. Goswami, A. M. Nzioki, S. D. Maharaj, S. G. Ghosh, Phys. Rev. D **90**, 084011 (2014).

[90] T. Harko, F. S. N. Lobo, Galaxies **2**, 410 (2014).

[91] T. Harko, Phys. Rev. D **81**, 044021 (2010).

[92] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).

[93] M. Sharif, M. Zubair, J. Phys. Soc. Jpn. **81**, 114005 (2012).

[94] F. G. Alvarenga, A. de la Cruz-Dombriz, M. J. S. Houndjo, M. E. Rodrigues, D. Sáez-Gómez, Phys. Rev. D **87**, 103526 (2013).

[95] J. O. Barrientos, G.F. Rubilar, Phys. Rev. D **90**, 028501 (2014).

[96] V. Faraoni, Phys. Rev. D **80**, 124040 (2009).

[97] H. Shabani, M. Farhoudi, Phys. Rev. D. **90**, 044031 (2014).

[98] M. Sharif, M. Zubair, JCAP **03**, 028 (2012).

[99] M. Jamil, D. Momeni, R. Myrzakulov, Chin. Phys. Lett. **29**, 109801 (2012).

[100] M. Sharif, M. Zubair, Gen. Relativ. Gravit., **46**, 1723 (2014).

[101] M. S. Houndjo, Int. J. Mod. Phys. D **21**, 1250003 (2012).

[102] Z. Yousaf, K. Bamba, M.Z. Bhatti, Phys. Rev. D **93**, 064059 (2016).

[103] Z. Yousaf, K. Bamba, M.Z. Bhatti, U. Ghafoor, Phys. Rev. D **100**, 024062 (2019).

[104] Z. Yousaf, Phys. Dark Univ. **28**, 100509 (2020).

[105] M. Z. Bhatti, Z. Yousaf, A. Rehman, Phys. Dark Univ. **29**, 100561 (2020).

[106] M. Sharif, A. Siddiqa, Eur. Phys. J. Plus **132**, 529 (2017).

[107] M. Sharif, A. Siddiqa, AHEP **(2019)**, 8702795 (2019).

[108] S. Khan, M. S. Khan, A. Ali, Modern Physics Letters A **33**, 1850065 (2018).

[109] M. Sharif, A. Anwar, Astrophys. Space Sci. **363**, 123 (2018).

[110] G. Abbas, R. Ahmed, Eur. Phys. J. C **77**, 441 (2017).

[111] R. Ahmed, G. Abbas, Can. J. Phys. **97**, 994 (2019).

[112] M. Zubair, H. Azmat, Phys. Dark Univ. **28**, 100531 (2020).

[113] Z. Yousaf, M.Z. Bhatti, H. Asad, Phys. Dark Univ. **28**, 100527 (2020).

[114] Z. Yousaf, M.Z. Bhatti, U. Farwa, Class. Quant. Gravity **34**, 145002 (2017).

[115] Z. Yousaf, M.Z. Bhatti, U. Farwa, Mon. Not. R Astron. Soc. **464**, 45094519 (2017).

[116] Z. Yousaf, K. Bamba, M.Z. Bhatti, U. Farwa, Eur. Phys. J. A **54**, 122 (2018).

[117] Z. Yousaf, Astrophys. Space Sci. **363**, 226 (2018).

[118] Z. Yousaf, Eur. Phys. J. Plus **134**, 245 (2019).

[119] Z. Yousaf, Mod. Phys. Lett. A **34**, 1950333 (2019).

[120] Z. Yousaf, Eur. Phys. J. Plus **136**, 281 (2021).

[121] C. W. Misner, D. Sharp, Phys. Rev. **136** B571 (1964).

[122] C. W. Misner, Phys. Rev. **137**, B1360, (1965).

[123] R. Tolman, Phys. Rev. **35**, 904 (1939).

[124] C. Eckart, Phys. Rev. **58**, 919 (1940).

[125] L. Landau and E. Lifshitz, Fluid Mechanics (Pergamon Press, London, 1959).

[126] I. Muller, Z. Physik **198**, 329 (1967).

[127] W. Israel, Ann. Phys. (N.Y.) **100**, 310 (1976).

[128] W. Israel, J. Stewart, Phys. Lett. A **58**, 213 (1976).

[129] W. Israel, J. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).

[130] L. Herrera, Int. J. Mod. Phys. D **15**, 2197 (2006).

[131] L. Herrera, J. Martinez, Astrophys. Space Sci. **259**, 235 (1998).

[132] L. Herrera, G. Le. Denmat, N. O. Santos, A. Wang, Int. J. Mod. Phys. D **13**, 583 (2004).

[133] L. Herrera, A. D. Prisco, E. Fuenmayor, O. Troconis, Int. Jour. of Mod. Phys. D **18**, 129, (2009).

[134] R. Maartens, Causal Thermodynamics in Relativity. Lectures given at the Hanno Rund Workshop on Relativity and Thermodynamics, University of Natal, June 1996. astro-ph/9609119.

[135] M. Sharif, Z. Rehmat, Gen Relativ Gravit **42**, 1795 (2010).

[136] S. Chakraborty, S. Chakraborty, AHEP **2017**, 8786791 (2017).

[137] S. M. Shah, G. Abbas, Eur. Phys. J. C **77**, 251 (2017).

[138] J. M. Z. Pretel, M. F. A. da Silva, Gen. Relat. Gravit. **51**, 3 (2019).

[139] M. Sharif, H. R. Kausar, Journal of Physics: Conference Series **354**, 012020 (2012).

[140] G. Abbas, H. Nazar, Adv. High Energy Phys. **2018**, 9250786 (2018).

[141] J.M.Z. Pretel, S.E. Jorás, R.R.R. Reis, J.D.V. Arba$\tilde{n}$il, (n.d.), JCAP **04**, 064 (2021); arXiv:2012.03342v1 [gr-qc]

[142] R. Ahmed, G. Abbas, Chinese J. Phys. **65**, 177 (2020).

[143] S. Thirukkanesh, S. S. Rajah, S. D. Maharaj, J. Math. Phys. **53**, 032506 (2012).

[144] These calculations were done with the help of Maple program and GRTensor package: K. Lake and P. J. Musgrave, GRTensor (Queen's University, Kingston, 2003), https://github.com/grtensor/grtensor

[145] S. Chandrasekhar, Astrophys. J. **140**, 417 (1964).

[146] L. Herrera and N.O. Santos, Phys. Rept. **286**, 53 (1997).

[147] B.K. Harrison, K.S. Thorne, J.A. Wheeler, *Gravitation Theory and Gravitational Collapse* (University of Chicago Press, Chicago, 1965).

[148] P. C. Vaidya, Nature, **171**, 260 (1953).

[149] S. Nojiri, S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003).

[150] D. E. Barraco, V. H. Hamity, Phys. Rev. D **62**, 044027 (2000).

[151] S. Capozziello, A. Stabile, A. Troisi, Class. Quantum Grav. **24**, 2153 (2007).

[152] S. Capozziello, A. Stabile, A. Troisi, Class. Quantum Grav. **25**, 085004 (2008).

[153] S. Capozziello, N. Frusciante, D. Vernieri, Gen. Relativ. Gravit. **44**, 1881 (2012).

[154] T. Multamaki, I. Vilja, Phys. Rev. D **74**, 064022 (2006).

[155] P. K. Sahoo, P. H. R. S. Moraes, Parbati Sahoo, B. K. Bishi, Eur. Phys. J. C **78**, 736 (2018).

[156] M. Sharif, M. Zubair, J. Phys. Soc. Jpn. **82**, 064001 (2013).

[157] C. P. Singh, V. Singh, Gen. Relativ. Gravit. **46**, 1696 (2014).

[158] P. H. R. S. Moraes, Eur. Phys. J. C **75**, 168 (2015).

[159] M. Sharif, M. Z. Gul, Mod. Phys. Lett. A **36**, 2150214 (2021).

[160] P. H. R. S. Moraes, P. K. Sahoo, Phys. Rev. D **96**, 044038 (2017).

[161] P. H. R. S. Moraes, P. K. Sahoo, Eur. Phys. J. C **77**, 480 (2017).

[162] P. H. R. S. Moraes, R. A. C. Correa, R. V. Lobato, JCAP **07**, 029 (2017).

[163] M. Sharif, A. Fatima, Eur. Phys. J. Plus **138**, 721 (2023).

[164] R. Zaregonbadi, M. Farhoudi, N. Riazi, Phys. Rev. D **94**, 084052 (2016).

[165] A. Wang, Y. Wu, Gen. Relativ. Gravit, **31**, 114 (1999).

[166] P. C. Vaidya, Proc. Indian Acad. Sci. (Math. Sci.) A, **33**, 264 (1951).

[167] M. Sharif, G. Abbas, J. Phys. Soc. Jpn **80**, 104002 (2011).

[168] P. S. Joshi, R. Goswami, N. Dadhich, Phys. Rev. D, **70**, 087502 (2004).

[169] A. Chatterjee, A. Ghosh, and S. C. Jaryal, Phys. Rev. D **102**, 064048 (2020).

[170] U. Debnath, S. Nath, S. Chakraborty, Mon. Not. R. Astron. Soc. **369**, 1961 (2006).

[171] S. W. Hawking, G. F. R. Ellis, *The Large Scale Structure Of Space-Time*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, (1973).

[172] G. Darmois, 'Les equations de la gravitation Einsteinienne,' *Mémorial de Sciences Mathématiques*, Fascicule XXV, ch V (1927).

[173] W. Israel, 'Singular Hypersurfaces and Thin Shells in General Relativity,' Nuovo Cimento **44**, 1 (erratum B **49** 463)(1966).

[174] N. Deruelle, M. Sasaki, Y. Sendouda, Prog. Theor. Phys. **119**, 237 (2008).

[175] T. Clifton, P. Dunsby, R. Goswami, A. M. Nzioki, Phys. Rev. D **87**, 063517 (2013).

[176] J. M. M. Senovilla, Phys. Rev. D **88**, 064015 (2013).

[177] R. Goswami, A. M. Nzioki, S. D. Maharaj, S. G. Ghosh, Phys. Rev. D **90**, 084011 (2014).

[178] Pankaj S. Joshi, *Gravitational Collapse and Spacetime Singularities*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, p. 67-68 (2007).

[179] P. S. Joshi, R. Goswami, N. Dadhich, arXiv:gr-qc/0308012v3 13 Jan 2004

[180] H. Azmat, M. Zubair, I. Noureen, Int. J. Mod. Phys. D, **26**, 1750181 (2017).

[181] Valeri. P. Frolov, Igor. D. Novikov, *Black Hole Physics - Basic Concepts and New Developments*, Fundamental Theories of Physics, Kluwer Academic Publishers, (1998), p. 17

[182] V. Faraoni, Galaxies **2013**, 1, 114-179

[183] P. Canate, L. G. Jaime, M. Salgado, Class. Quantum Grav. **33**, 155005 (2016).

[184] B. Hazarika, P. Phukon, Front Phys. **20**, 035201 (2025); arXiv:2410.00606v2 [gr-qc] 2 Oct 2024

[185] M. Dutta, B. Mukhopadhyay, arXiv:2411.18693v1 [gr-qc] 27 Nov 2024

[186] L. Herrera, A. Di Prisco, J. Ibanez, Phys. Rev. D **84**, 107501 (2011).

[187] L. Herrera, A. Di Prisco, J. Ospino, Gen. Relativ. Gravit. **44**, 2645 (2012).

[188] M. Sharif, M. Bhatti, Gen. Relativ. Gravit. **44**, 2811 (2012).

[189] M. Sharif, M. Bhatti, Mod. Phys. Lett. A, Vol. **27**, 1250141 (2012).

[190] M. Sharif, M. Bhatti, Astrophys. Space Sci. **349**, 995 (2014).

[191] M. Sharif, M. Bhatti, Int. J. Mod. Phys. D, Vol. **24**, 1550014 (2015).

[192] M. Sharif, R. Manzoor, Phys. Rev. D **91**, 024018 (2015).

[193] M. Sharif, R. Manzoor, Astrophys. Space Sci. **359**, 17 (2015).

[194] M. Sharif, Z. Yousaf, Astrophys. Space Sci. **357**, 49 (2015).

[195] M. Z. Bhatti, Z. Yousaf, Z. Tariq, Eur. Phys. J. C **81**, 16 (2021).

[196] M. Z. Bhatti, Z. Yousaf, Z. Tariq, Chinese J. Phys. **72**, 18 (2021).

[197] M. Z. Bhatti, Z. Yousaf, Z. Tariq, Phys. Scr. **96**, 115301 (2021).

[198] U. Ghosh, and S. Guha, **Gen Relativ Gravit 57 , 47 (2025)**; DOI : **10.1007/s10714-025-03379-0**.

[199] T. Hussain, M. Khurshudyan, S. Ahmed, As. Khurshudyan, Int. J. Mod. Phys. D, Vol. **26**, 1750155 (2017).

[200] Z. Yousaf, M.Z. Bhatti, U. Farwa, Ann. Phys. (2021), doi: https://doi.org/10.1016/j.aop.2021.168601.

[201] Z. Yousaf, M. Z. Bhatti, T. Naseer, Eur. Phys. J. Plus **135**, 323 (2020).

[202] U. Ghosh, S. Guha, **Phys. Lett. B 869 , 139834 (2025)**; DOI : **10.1016/j.physletb.2025.139834**.

[203] M. Sharif, M. Zubair, J. Phys. Soc. Jpn **82**, 014002 (2013).

[204] L. Herrera, A. Di Prisco, and J. Ibanez, Phys. Rev. D **84**, 107501 (2011).

[205] L. Herrera, A. Di Prisco, J. Ospino, Gen Relativ Gravit **44**, 2645 (2012).

[206] M. Sharif, and M. Bhatti, Gen Relativ Gravit **44**, 2811 (2012).

[207] M. Sharif, and M. Bhatti, Mod. Phys. Lett. A, Vol. **27**, 1250141 (2012).

[208] M. Sharif, and M. Bhatti, Astrophys Space Sci **349**, 995 (2014).

[209] M. Sharif, and M. Bhatti, Int. Journal of Mod. Phys. D, Vol. **24**, 1550014 (2015).

[210] M. Sharif, and R. Manzoor, Phys. Rev. D **91**, 024018 (2015).

[211] M. Sharif, and R. Manzoor, Astrophys. Space Sci. **359**, 17 (2015).

[212] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rept. **692**, 1 (2017).

[213] A. V. Astashenok, K. Mosani, S. D. Odintsov and G. C. Samanta, Int. J. Geom. Meth. Mod. Phys. **16**, 1950035 (2019).

[214] M. Sharif, Z. Yousaf, Astrophys. Space Sci. **357**, 49 (2015).

[215] M. Z. Bhatti, Z. Yousaf, Z. Tariq, Eur. Phys. J. C **81**, 16 (2021).

[216] M. Z. Bhatti, Z. Yousaf, Z. Tariq, Phys. Scr. **96**, 115301 (2021).

[217] M. Z. Bhatti, Z. Yousaf, Z. Tariq, Chinese Journal of Physics **72**, 18 (2021).

[218] Z. Yousaf, K. Bamba, and M. Z. Bhatti, Phys. Rev. D **93**, 064059 (2016).

[219] Z. Yousaf, K. Bamba, and M. Z. Bhatti, Phys. Rev. D **93**, 124048 (2016).

[220] T. Hussain, M. Khurshudyan, S. Ahmed, As. Khurshudyan, Int. Journal of Mod. Phys. D, **26**, 1750155 (2017).

[221] Z. Yousaf, M.Z. Bhatti and U. Farwa, Ann. Phys. **433**, 168601 (2021).

[222] L. Herrera, Phys. Rev. D **97**, 044010 (2018).

[223] L. Herrera, A. Di Prisco, and J. Ospino, Phys. Rev. D **98**, 104059 (2018).

[224] L. Herrera, A. Di Prisco, and J. Ospino, Phys. Rev. D **99**, 044049 (2019).

[225] L. Herrera, A. Di Prisco, and J. Carot, Phys. Rev. D **99**, 124028 (2019).

[226] M. Sharif, and I. I. Butt, Eur. Phys. J. C **78**, 688 (2018).

[227] M. Sharif and S. Tariq, Mod. Phys. Lett. A **35**, 2050231 (2020).

[228] Z. Yousaf, M. Z. Bhatti, and T. Naseer, Phys. of the Dark Univ. **28**, 100535 (2020).

[229] J. S. Gagnon and J. Lesgourgues, Journ. of Cosmo. and Astroparticle Phys. **09**, 026 (2011).

[230] J. L. Rosa, Phys. Rev. D **103**, 104069 (2021).

[231] C. A. Kolassis, N. O. Santos, and D. Tsoubelis, Class. Quantum Grav. **5**, 1329 (1988).

[232] A.V.Astashenok, S.Capozziello, S.D.Odintsov, V.K.Oikonomou, Phys. Lett. B **811**, 135910 (2020).

[233] A.V.Astashenok, S.Capozziello, S.D.Odintsov, V.K.Oikonomou, Phys. Lett. B **816**, 136222 (2021).

[234] S. A. Mardan, Maha, R. Manzoor, M. B. Riaz, Eur. Phys. J. Plus **140**, 85 (2025).

[235] M. Sharif, H. R. Kausar, Astrophys. Space Sci. **331**, 281 (2011).

[236] A. Casado-Turrión, A. Dobado, A. dela Cruz-Dombriz, Phys. Rev. D **105**, 084060 (2022).